The metric for non-Hermitian Hamiltonians: a case study

By

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Thesis presented in partial fulfilment of the requirements for the degree of
MASTER OF SCIENCE at the University of Stellenbosch

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December 2006

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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Signature Date

Abstract

We are studying a possible implementation of an appropriate framework for a proper non-Hermitian quantum theory. We present the case where for a non-Hermitian Hamiltonian with real eigenvalues, we define a new inner product on the Hilbert space with respect to which the non-Hermitian Hamiltonian is Quasi-Hermitian. The Quasi-hermiticity of the Hamiltonian introduces the bi-orthogonality between the left-hand eigenstates and the right-hand eigenstates, in which case the metric becomes a basis transformation. We use the non-Hermitian quadratic Hamiltonian to show that such a metric is not unique but can be uniquely defined by requiring to hermitize all elements of one of the irreducible sets defined on the set of all observables. We compare the constructed metric with specific known examples in the literature in which cases a unique choice is made.

Opsomming

Ons ondersoek die implementering van n gepaste raamwerk virn nie-Hermitiese kwantumteorie. Ons beskoun nie-Hermitiese Hamilton-operator met reele eiewaardes en definieer in gepaste binneproduk ten opsigte waarvan die operator kwasi-Hermitiese is. Die kwasi- Hermities aard van die Hamilton operator lei dan tot n stel bi-ortogonale toestande. Ons konstrueer n basistransformasie wat die linker en regter eietoestande van hierdie stel koppel. Hierdie transformasie word dan gebruik omn nuwe binneproduk op die Hilbert-ruimte te definieer. Die oorspronklike nie-HermitieseHamilton-operator is danHermitiesmet betrekking tot hierdie nuwe binneproduk. Ons gebruik die nie-Hermitiese kwadratieseHamilton-operator om te toon dat hierdie metriek nie uniek is nie, maar wel uniek bepaal kan word deur verder te vereis dat dit al die elemente van n onherleibare versameling operatoreHermitiseer. Ons vergelyk hierdie konstruksie met die bekende voorbeelde in die literatuur en toon dat die metriek in beide gevalle uniek bepaal kan word.

To my son Dibwe Elijah Musumbu.

Acknowledgements

I wish to express my deep gratitude to
Professor H.B. Geyer and Professor W.D. Heiss from the Institute of Theoretical Physics within the Physics Department at the University of Stellenbosch, who supervised me and proposed this topic.

I also gratefully acknowledge the generous financial support received from the NRF, the Department of Physics, the Dean of Sciences Faculty and the African Institute for Mathematical Sciences while writing this thesis.

I thank also Professor F.G. Scholtz and Doctor Miloslav Znojil for collaboration and valuable discussions, and Izak Snyman, Hannes Kriel, Henry Amuasi and Lee Bonzaaier for valuable discussions and support in the writings.

I thank Professor Philippe Badibanga Mudibu for his support, and all my colleagues and the staff for the flows of joy, fun, and jokes and every other thing that made my life easy and fun here in Stellenbosch.

Special thanks to my Lord Jesus-Christ for being permanently present.

CONTENTS

Introduction

This thesis is concerned with the idea that the Quasi-hermiticity represents an alternative to the axiom of hermiticity in quantum mechanics[1]. Quantum mechanics is a framework extending the fundamental physical theory described by Newtonian mechanics at the atomic and subatomic level. This framework has proved to be very successful in many branches of physics, providing accurate and precise descriptions of many phenomena for which Newtonian mechanics breaks down. The predictions of quantum mechanics have been validated by a centurys worth of experiments. Even though the term quantum[1] refers to the discrete units that the theory assigns to certain physical quantities, quantum mechanics also forms the basis for descriptions of phenomenon like wave-particle duality and quantum entanglement. In general, physics is based on the scientific method where the observation plays a major role in the description of a phenomenon, and mechanics is a branch of physics observing physical quantities such as energy and momentum to study the motion of bodies. The physical observables relate in mathematical formulation of quantum mechanics to linear operators and theirs projections on Hilbert space.

Mainly, there exist two formulations of quantum mechanics namely *matrix quantum mechanics* and *wave quantum mechanics* which are unified in Dirac's formulation of quantum mechanics[8]. This formulation deals with the framework presenting quantum mechanics through mappings on the Hilbert space. In this, the quantum mechanical descriptions turn out to become a set of linear operators associated with physical quantities called observables and the Hilbert space on which they act. This set of observables finds its foundation in a so called quantization procedure building quantum mechanics from classical mechanics. With the development of quantum theory the quantization has been represented and is formulated by commutators. The commutators are consequences of Poisson bracket and the Heisenberg uncertainty. In Diracs formulation of quantum mechanics, the commutators represent fundamental quantum conditions. In quantum mechanics the Hilbert space represents the ensemble of all physical states of the system. The fact that the observables position $\hat{\mathbf{x}}$ and momentum $\hat{\mathbf{p}}$ do not commute with each other implies that in the Hilbert space there is not a physical state which is a simultaneous eigenstate of the position $\hat{\mathbf{x}}$ and momentum $\hat{\mathbf{p}}$. Such a fact is a consequence of the Heisenberg uncertainty relation.

In the standard interpretation of quantum mechanics the value of an observable $\hat{\mathbf{A}}$ is sharply measured if the system is in its eigenstate. The expected outcome of the measurement is the

[1] The word quantum in Latin means "how much".

eigenvalue of this eigenstate. The expected outcome of the measurement is the eigenvalue of this eigenstate. Mathematically speaking the eigenstate may be considered as a vector in an infinite-dimensional complex space. The commutation requirement plays a crucial role in quantum mechanics since for a complete quantum description of a given system, we need to have a set of Hermitian commuting observables and a Hilbert space. The hermiticity of observables guarantees real expectations values in the study of the system. The hermiticity criteria ensures that the eigenvalues and expectation values of physical observables are real. Algebraically the hermiticity of an observable may be interpreted as not only the extension of the set of commutative observables to their Hermitian conjugates[32], but also the equality between an operator and its Hermitian conjugate.

On the other side, even though diverse physical applications are related to non-Hermitian Hamiltonians in such a diverse area as ionization optics, transitions in superconductors, dissipative quantum systems, quantum cosmology, etc..., we are only interested in the unitary non-Hermitian systems. So far the study of these non-Hermitian cases appear itself interesting in that some unexpected properties and details lost in the Hermitian considerations may show up in the non-Hermitian considerations. The use of non-Hermitian operators has revealed itself more efficiently in many illustrations. As pointed out by [1] in the use of the mapping of Hermitian fermion operators onto non-Hermitian bosons operators. The choice of Hermitian observables in Hilbert space fits the need of a complete mathematical framework in quantum theory, it appears interesting to consider an alternative offering a comparative observation. In stepping out of the standard concept of hermiticity [1], we open the quantum theory to new considerations on the mathematical structures of the quantum mechanics. Straightforwardly, the most important thing in quantum mechanics is the reality of eigenvalues and expectation values. In that direction there is more hope coming from Quasi-Hermitian Hamiltonians [3][6][10]. These Hamiltonians can have real spectrum, and are suitable for the reformulation that removes the constraints. In this reformulation, the picture requiring the self-adjoint observables becomes a particular case and the framework opens quantum mechanics to a larger class of observables involving both Hermitian and Quasi-Hermitian observables (Mostly Quasi-Hermitian Hamiltonians). One may ask why do non-Hermitian Quasi-Hermitian Hamiltonians have a such particularity? Intuitively this is explained by the presence of the symmetry involving some conservation. Starting from the genesis of the standard quantum mechanics, the hermiticity requirement fits the use of quantum mechanical unbounded observables in a mathematical framework. Since the idea coming from the $\mathcal{CPT}$ theorem follows from the axioms of local quantum field theory[19], there has been an increasing interest for non-Hermitian Hamiltonians, and several works on more physical alternatives to the hermiticity requirements and the *space-time reflection symmetry* ($\mathcal{PT}$ -symmetry) is successful.

This thesis is organized in six chapters. We introduce the first by drawing parallelism between some background functional analysis concepts and the non-Hermitian quantum mechanics philosophy. In the second chapter we briefly address the general considerations of the construction of the mapping performing the conversion of a non-Hermitian operator into a Hermitian operator and how all this can fit into a metric framework by illustrating the case using the non-Hermitian two bosons quadratic Hamiltonian. In the third chapter, we address the theoretical background of the diagonalization via Bogoliubov transformation. In the fourth chapter we address the problem of measurement as it appears through both pictures. We discuss the uniqueness problem and construct the link between the two methods. In the fifth chapter we are analyzing the physical aspects of the solutions in addressing the problem of the eigenstates of the two bosons quadratic non-Hermitian Hamiltonian. We close this chapter with illustrating some practical physics cases. In the conclusion we present the results and further observations on the framework of the non-Hermitian quantum mechanics.

CHAPTER 1
MATHEMATICAL FOUNDATIONS

1.1 Definitions

This chapter presents the basic notions underlining concepts such as *Linear Operators* on *Hilbert Spaces*. They are fundamental ingredients in the formulation of quantum mechanics. As it appears today, quantum mechanics provides the most accurate and complete description of the world yet discovered. Its mathematical formulation is characterized by the use of abstract mathematical structures, such as Hilbert spaces and operators on these spaces. In quantum mechanics, physical quantities such as energy and momentum are no longer represented by functions on some phase space, but rather by operators acting on a Hilbert space. These functions represent the states of the physical system and form the Hilbert space. Physical quantities in quantum mechanics formalism are represented by the expectation values of linear operators. Therefore quantum mechanics formalism is built on the juxtaposition of two of mathematical domains. We define all these basic concepts and all related notions. These definitions set a good understanding on the use of these two concepts in this thesis. Most of these definitions come from the three references[13], [14], and [15]. The Hilbert space is basically sitting on the intersection of the *metric space* and the *vectors space*.

Definition 1.1.1 *A metric space is a pair* $(\mathscr{M}, d)$ *where* $\mathscr{M}$ *is a set and* d *is a metric on* $\mathscr{M}$ *(or a distance function on* $\mathscr{M} \times \mathscr{M}$*) such that for all* x, y *and* z *in* $\mathscr{M}$ *we have:*

d is real valued, finite and non-negative.

$d(x, y) = 0$ *if and only if* $x = y$*.*

$d(x, y) = d(y, x)$ *(Symmetry),*

$d(x, y) \leq d(x, z) + d(z, y)$ *(triangle equality).*

The metric function creates a natural setting which fixes the *closeness*[2] of points in a metric space. This idea is essential in the study of a sequence.

Definition 1.1.2 *A sequence is called a Cauchy sequence if the terms of the sequence eventually all become arbitrarily close to one another. In a metric space* $\mathscr{M}$ *with a metric* d*, a Cauchy sequence is such that for every positive real number* r*, there is an integer* N *such that for all integers* $m, n > N$ *the metric* $d(x_m, x_n)$ *is less than* r*.*

[2] The relative position of each point with respect to another

Definition 1.1.3 *A metric space $\mathscr{M}$ is said to be complete if every Cauchy sequence defined on $\mathscr{M}$ converges in $\mathscr{M}$.*

Definition 1.1.4 *A subset $\mathscr{N}$ of a metric space $\mathscr{M}$ is dense in $\mathscr{M}$ if each point in $\mathscr{N}$ is infinitesimally close to at least one point in $\mathscr{M}$.*

Definition 1.1.5 *A vector space $\mathscr{V}$ over [3] $\mathbb{C}$ is the set of object denoted by [4] $|\phi\rangle$, $|\varphi\rangle$ and $|\psi\rangle$,..., called vectors, with the following properties:*

* *To every pair of vectors $|\phi\rangle$, and $|\varphi\rangle$ corresponds a vector $|\phi\rangle + |\varphi\rangle$, also in $\mathscr{V}$, called sum of $|\phi\rangle$, and $|\varphi\rangle$, such that*

(*a*) $|\phi\rangle + |\varphi\rangle = |\varphi\rangle + |\phi\rangle$,

(*b*) $|\phi\rangle + (|\varphi\rangle + |\psi\rangle) = (|\phi\rangle + |\varphi\rangle) + |\psi\rangle$,

(*c*) *There exists a unique vector $|0\rangle$ in $\mathscr{V}$, called the zero vector, such that $|\phi\rangle + |0\rangle = |\phi\rangle$ for every vector $|\phi\rangle$,*

(*d*) *To every vector $|\phi\rangle$ in $\mathscr{V}$ there correspond a unique vector $-|\phi\rangle$ (also in $\mathscr{V}$) such that $|\phi\rangle + (-|\phi\rangle) = |0\rangle$.*

* *To every complex scalar λ and every vector $|\phi\rangle$ there corresponds a vector $\lambda|\phi\rangle$ in $\mathscr{V}$ such that*

(*a*) $\lambda(\nu|\phi\rangle) = (\lambda\nu)|\phi\rangle$, *with ν a complex number.*

(*b*) $1|\phi\rangle = |\phi\rangle$.

* *Multiplication involving vectors and scalars is distributive:*

(*a*) $\lambda(|\phi\rangle + |\varphi\rangle) = \lambda|\phi\rangle + \lambda|\varphi\rangle$,

(*b*) $(\lambda + \nu)|\phi\rangle = \lambda|\phi\rangle + \nu|\varphi\rangle$.

When $\mathscr{V}$ is over $\mathbb{R}$ the vector space is said to be a real vector space.

Definition 1.1.6 *The vectors $|\varphi_i\rangle$ (for $i = 1, ...n$), are said to be linearly independent if for complex scalars λ_i the linear combination $\sum_{i=1}^{n} \lambda_i|\varphi_i\rangle = 0$ implies $\lambda_i = 0$ for all i.*

Definition 1.1.7 *A dual space $\mathscr{V}^*$ of a real vector space $\mathscr{V}$ is the set of linear functional on $\mathscr{V}$. In Dirac notation $\langle\varphi_j|$ are linear functionals of $|\varphi_j\rangle$.*

Definition 1.1.8 *A subspace $\mathscr{V}_\mu$ of a vector space $\mathscr{V}$ is a nonempty subset of $\mathscr{V}$ with the property that if $|\varphi_i\rangle, |\varphi_j\rangle$ belong to $\mathscr{V}_\mu$, the sum $\lambda_k|\varphi_i\rangle + \lambda_l|\varphi_j\rangle$ belong also to $\mathscr{V}_\mu$ for all complex numbers λ_k and λ_l.*

[3] with $\mathbb{C}$ set of complex numbers

[4] We use the Dirac *bra*, *ket* notation for vectors.

Definition 1.1.9 *A basis of a vector space $\mathscr{V}$ is a set $\mathscr{B}$ of linearly independent vectors that spans all of $\mathscr{V}$. A vector space that have a finite basis is called finite dimensional vector space; otherwise it is an infinite dimensional vector space.*

Definition 1.1.10 *An inner product of two vectors $|\varphi_i\rangle$, and $|\varphi_j\rangle$ in a vector space $\mathscr{V}$ is a complex number $\langle\varphi_i|\varphi_j\rangle$, such that*

(1) $\langle\varphi_i|\varphi_j\rangle = \langle\varphi_j|\varphi_i\rangle^*$,

(2) $\langle\varphi_i|(\lambda_m|\varphi_j\rangle + \lambda_n|\varphi_k\rangle) = \lambda_m\langle\varphi_i|\varphi_j\rangle + \lambda_n\langle\varphi_i|\varphi_k\rangle$,

(3) $\langle\varphi_i|\varphi_i\rangle \geq 0$ *and* $\langle\varphi_i|\varphi_i\rangle = 0$ *if and only if* $|\varphi_i\rangle = |0\rangle$.

A vector space on which an inner product is defined is called an inner product space. Consequently, all finite dimensional vector spaces can be turned into inner product spaces.

Definition 1.1.11 *A Hilbert space is complete inner product space. In this thesis we will denote a Hilbert space by $\mathscr{H}$. A Hilbert space which has a countable dense subset is said to be separable.*

The Hilbert space $\mathscr{H}$ and its dual conjugate $\mathscr{H}^*$ are always isomorphic.

Definition 1.1.12 *The norm, or length, of a vector $|\varphi_i\rangle$ in an inner product space is the real number given by $\sqrt{\langle\varphi_i|\varphi_i\rangle}$.*

The linear operators represents physical observables. Their expectation values are the physical quantities directly measurable on the physical system.

Definition 1.1.13 *A linear operator from the complex vector space $\mathscr{U}$ to the complex vector space $\mathscr{V}$ is a mapping*

$\hat{\mathbf{O}} : \mathscr{U} \longrightarrow \mathscr{V}$ *such that*

$$\hat{\mathbf{O}}(\lambda_m|\varphi_j\rangle + \lambda_n|\varphi_k\rangle) = \lambda_m\hat{\mathbf{O}}|\varphi_j\rangle + \lambda_n\hat{\mathbf{O}}|\varphi_k\rangle$$

where all $|\varphi_i\rangle$ are in $\mathscr{U}$ and all λ_r are complex numbers.

When $\mathscr{U} = \mathscr{V}$, the linear operator is said linear operator on $\mathscr{U}$. and when $\mathscr{V} = \mathbb{C}$ the linear operator is called **linear functional**[5].

From now all operators we will use in this thesis are linear. Therefore when referring to linear operators we will be using simply "operator".

Definition 1.1.14 *The inverse of a linear operator $\hat{\mathbf{O}}$ is a linear operator denoted $\hat{\mathbf{O}}^{-1}$ such that*

$$\hat{\mathbf{O}}^{-1}\hat{\mathbf{O}} = \hat{\mathbf{O}}\hat{\mathbf{O}}^{-1} = \mathbf{I}.$$

[5]This holds also for $\mathscr{V} = \mathbb{R}$

Definition 1.1.15 *An operator* $\hat{\mathbf{B}}$ *defined on a Hilbert space* $\mathscr{H}$ *with domain* $\mathscr{D}(\hat{\mathbf{B}}) \subseteq \mathscr{H}$ *is said to be* bounded *if there exists a positive real number* r *such that* $\hat{\mathbf{B}}$ *maps all vectors* $|\varphi_j\rangle$ *inside a finite shell of "radius"* r *in the Hilbert space* $\mathscr{H}$*. in other words* $\langle\varphi_j|\hat{\mathbf{B}}|\varphi_j\rangle \leq r\langle\varphi_j|\varphi_j\rangle$*, for* $|\varphi_j\rangle \in \mathscr{D}(\hat{\mathbf{B}})$ *and* r *real and positive.*

Definition 1.1.16 *A Hilbert space adjoint* [6] *of* $\hat{\mathbf{B}}$ *(or simply Hermitian conjugate of* $\hat{\mathbf{B}}$ *) denoted* $\hat{\mathbf{B}}^\dagger$ *is the linear operator on* $\mathscr{H}$*. For all* $|\psi\rangle$ *in its domain* $\mathscr{D}(\hat{\mathbf{B}})$*, and all* $|\varphi\rangle$ *in its range* $\mathscr{R}(\hat{\mathbf{B}})$*, there exists* $|v\rangle$ *with* $\hat{\mathbf{B}}^\dagger|\varphi\rangle = |v\rangle$ *such that*[7]

$$\langle\psi|\hat{\mathbf{B}}|\varphi\rangle = (\langle\varphi|\hat{\mathbf{B}}^\dagger|\psi\rangle)^*.$$

When $\hat{\mathbf{B}}^\dagger = \hat{\mathbf{B}}$ *the operator* $\hat{\mathbf{B}}$ *is said 'self-adjoint'.*

Many of the most important operators which are used in quantum mechanics are unbounded operators. The *Hellinger-Toeplitz theorem*([14] page 525) shows that if a self-adjoint operator is defined on whole of a Hilbert space $\mathscr{D}(\hat{\mathbf{B}}) = \mathscr{H}$, then it must be bounded. This result is very helpful in the understanding *unbounded operators* $\hat{\mathbf{U}}$ since it shows that an unbounded self-adjoint operator cannot not be defined on all of the Hilbert space containing its domain i.e. $\mathscr{D}(\hat{\mathbf{U}}) \neq \mathscr{H}$; it can only be defined on a dense subset of $\mathscr{H}$.

Definition 1.1.17 *Let* $\hat{\mathbf{U}}$ *be a densely defined operator on a Hilbert space* $\mathscr{H}$*. An operator* $\hat{\mathbf{U}}^\dagger$ *is called the Hermitian conjugate of* $\hat{\mathbf{U}}$*, if there is a domain* $\mathscr{D}(\hat{\mathbf{U}}^\dagger)$ *set of* $|\varphi\rangle \in \mathscr{H}$ *for which there is an* $|v\rangle \in \mathscr{H}$ *with*

$$\langle\varphi|\hat{\mathbf{U}}|\psi\rangle = \langle v|\psi\rangle \text{ for all } |\psi\rangle \in \mathscr{D}(\hat{\mathbf{U}}).$$

For each of such $|\varphi\rangle \in \mathscr{D}(\hat{\mathbf{U}}^\dagger)$*, we define* $\hat{\mathbf{U}}^\dagger|\varphi\rangle = |v\rangle$*.*

Definition 1.1.18 *A densely defined operator* $\hat{\mathbf{U}}$ *on a Hilbert space* $\mathscr{H}$ *is called Hermitian if* $\mathscr{D}(\hat{\mathbf{U}}) \subset \mathscr{D}(\hat{\mathbf{U}}^\dagger)$ *and* $\hat{\mathbf{U}}|\psi\rangle = \hat{\mathbf{U}}^\dagger|\psi\rangle$ *for all* $|\psi\rangle \in \mathscr{D}(\hat{\mathbf{U}})$*.*

Here comes the separation between self-adjointness and hermiticity; for unbounded operators. The domain $\mathscr{D}(\hat{\mathbf{U}})$ is not necessarily equal to the domain $\mathscr{D}(\hat{\mathbf{U}}^\dagger)$ for the hermiticity[32]. While the self-adjointness requires that $\mathscr{D}(\hat{\mathbf{U}}) = \mathscr{D}(\hat{\mathbf{U}}^\dagger)$.

Definition 1.1.19 *(1) Let* $\mathscr{V}_\pm$ *two inner product spaces endowed with Hermitian linear automorphisms[10]* $\mathbf{M}_\pm$ *(inve operators mapping* $\mathscr{V}_\pm$ *to itself and satisfying)*

For all $v_\pm$*,* $w_\pm$ *in* $\mathscr{V}_\pm$*,*

$$(v_\pm, \mathbf{M}_\pm w_\pm)_\pm = (\mathbf{M}_\pm v_\pm, w_\pm)_\pm,$$

[6]For the bounded operators Hilbert space adjoint and the Hermitian conjugate are the same concepts.

[7]We recall that a Hilbert space $\mathscr{H}$ and its dual conjugate $\mathscr{H}^*$ are isomorphic.

(where $(\ ,\ )_{\pm}$ *stands for inner product of* $\mathscr{V}_{\pm}$*) and* $\mathbf{O} : \mathscr{V}_{+} \longrightarrow \mathscr{V}_{-}$ *be a linear operator. Then the pseudo-Hermitian adjoint* $\mathbf{O}^{\ddagger} : \mathscr{V}_{-} \longrightarrow \mathscr{V}_{+}$ *of* $\mathbf{O}$ *is defined by* $\mathbf{O}^{\ddagger} = \mathbf{M}_{+}^{-1}\mathbf{O}\mathbf{M}_{-}$.

In particular for $\mathscr{V}_{\pm} = \mathscr{V}$ *and* $\mathbf{M}_{\pm} = \mathbf{M}$*, the operator O is said to be* $\mathbf{M}_{\pm}$*-pseudo-Hermitian if* $\mathbf{O}^{\ddagger} = \mathbf{O}$.

(2) Let V be two inner product spaces. Then a linear operator $\mathbf{O} : \mathscr{V} \longrightarrow \mathscr{V}$ *is said to be pseudo-Hermitian, if there is a Hermitian linear automorphism* $\mathbf{M}$ *such that* $\mathbf{O}$ *is M-pseudo-Hermitian.*

1.2 The non-Hermitian quantum mechanics philosophy

Standard quantum mechanics substitutes the classical idea of the states variables (physical quantities such as energy, momentum, position, etc...) as functions defined on the phase space by the idea of linear operators (physical observables such as the Hamiltonian operator, the particles number operator, etc...) acting on a Hilbert space. Among all these linear operator, standard Quantum mechanics focuses on the Hamiltonian since it incorporates all the symmetries of the theory. In addition the theory need be: measurable, probability conserving these two requirements impose only the use of *Hermitian* observables (or linear operators). In other words these requirements are: The *unitarity* of the time evolution; which ensures the conservation of probability, and the *reality* of expectation values and eigenvalues which ensures the measurability of the theory. In the previous section we have seen that the properties of an operator are strongly related to the properties of the Hilbert space on which it acts, as we shall see, the hermiticity of the Hamiltonian imposes only the use of real Hilbert spaces with positive definite inner product. The fact that an amazingly precise theory of physics is built on purely abstract mathematical criteria, stems from the combination of mathematical logic and the strength of physical insights. How far can quantum mechanics bring us if we can relax these restrictions while keeping mathematical logic and physical insights. This idea is the foundation of the $\mathcal{PT}$-symmetric quantum mechanics. $\mathcal{PT}$-symmetric quantum mechanics attempts to find a framework substituting the Hermitian Hamiltonians and the real Hilbert spaces by more physical requirements which allows the extension of quantum theory to a much larger class of non-Hermitian Hamiltonians that are $\mathcal{PT}$-symmetric. The main challenge of $\mathcal{PT}$-symmetric quantum mechanics is to ensure that it retains the key physical properties that quantum mechanics exhibits.

The main reason why Hermitian observables are chosen in the description of quantum theory is that it guarantees the reality of the spectrum. Additionally, the association that standard quantum mechanics provides between the states in mathematical Hilbert spaces and experimentally measurable probabilities requires the use of positive definite real Hilbert space. Therefore, the most important question would be: *What are the necessary and sufficient conditions for the reality of the spectrum of a linear operator?* A. Mostafazadeh and A. Batal in [10] have shown

that if a linear operator acting on a Hilbert space $\mathscr{H}$ has a complete set of eigenvectors (i.e it is diagonalizable) then its spectrum is real if and only if the following equivalent conditions holds:
$c1$ There exists a positive-definite operator $\eta_+ : \mathscr{H} \to \mathscr{H}$ that fulfils

$$\mathbf{H}^{\dagger} = \eta_+ \mathbf{H} \eta_+^{-1} \tag{1.1}$$

$c2$ $\mathbf{H}$ is Hermitian with respect to some positive-definite inner product on $\mathscr{H}$; In i.e., if the normal inner product is defined as $\langle \varphi_j | \varphi_k \rangle$, we denote the positive definite inner product as, $\langle \varphi_j | \varphi_k \rangle_+ \geq 0$. A specific choice is provided by $\langle \varphi_j | \eta_+ | \varphi_k \rangle$.
$c3$ $\mathbf{H}$ may be mapped to a Hermitian Hamiltonian $\tilde{\mathbf{H}}$ by a similarity transformation.
Such a Hamiltonian is said to be *Pseudo Hermitian* with respect to the positive definite inner product $\langle \varphi_j | \eta_+ | \varphi_k \rangle$. The operator η_+ is bounded, Hermitian and invertible and defined on the entire space $\mathscr{H}$: Such an operator is called a *metric*, since it is used to define the so called η *-inner product*.

On the other hand, let us consider an *anti-Hermitian* operator τ :

$$\langle \varphi_j | \tau | \varphi_k \rangle = \langle \varphi_k | \tau | \varphi_j \rangle * . \tag{1.2}$$

The Hamiltonian $\mathbf{H}$ is said to be *pseudo-anti-Hermitian* with respect to τ if

$$\mathbf{H}^{\dagger} = \tau \mathbf{H} \tau^{-1}. \tag{1.3}$$

The operator τ is unique up to basis transformations. Even though the equation (1.1) and (1.3) are similar, the transformation η and τ are conceptually different. The first serves in the requirements for the observable $\mathbf{H}$ to be diagonalizable while the second; τ defines the pseudo-anti-Hermiticity.
A Mostafazadeh in [10, 11] shows that every pseudo-anti Hermitian with respect to an operator τ is diagonalizable. Therefore, the Hamiltonian $\mathbf{H}$ is both pseudo-Hermitian with respect to η_+ ,and pseudo-anti-Hermitian τ. Consequently, $\mathbf{H}$ admits an *anti-linear-symmetry* χ such that

$$\chi = \eta_+ \tau. \tag{1.4}$$

This shows that *every diagonalizable pseudo-Hermitian Hamiltonian* $\mathbf{H}$ *admits an anti-linear symmetry.*

In Standard quantum theory we learn that the Hamiltonian incorporates two kinds of symmetries; continuous symmetries, such as Lorentz transformations and discrete symmetries, such

as charge conjugation, parity invariance and time reversal invariance. Since we are looking at non-relativistic quantum mechanics we are interested in the parity (or space reflection) symmetry $\mathcal{P}$ and the time reversal symmetry $\mathcal{T}$. These two symmetries are defined with respect to their respective actions on the position operator $\hat{\mathbf{x}}$ and the momentum operator $\hat{\mathbf{p}}$; in i.e.:
The parity operator $\mathcal{P}$ is linear,

$$\begin{aligned} \mathcal{P}\hat{\mathbf{x}}\mathcal{P}^{-1} &= -\hat{\mathbf{x}} && (1.5)\\ \mathcal{P}\hat{\mathbf{p}}\mathcal{P}^{-1} &= -\hat{\mathbf{p}} && (1.6)\\ \mathcal{P}(i\mathbf{I})\mathcal{P}^{-1} &= i\mathbf{I}, && (1.7) \end{aligned}$$

and the time reversal operator $\mathcal{T}$ is anti-linear, it acts such that

$$\begin{aligned} \mathcal{T}\hat{\mathbf{x}}\mathcal{T}^{-1} &= \hat{\mathbf{x}} && (1.8)\\ \mathcal{T}\hat{\mathbf{p}}\mathcal{T}^{-1} &= -\hat{\mathbf{p}} && (1.9)\\ \mathcal{T}(i\mathbf{I})\mathcal{T}^{-1} &= -i\mathbf{I}. && (1.10) \end{aligned}$$

Equation (1.10) is required to preserve the fundamental commutation relation $[\hat{\mathbf{x}}, \hat{\mathbf{p}}] = i$. We can see that the combination of these two symmetries is an anti-linear symmetry.

Let us consider a non-Hermitian Hamiltonian $\mathbf{H} \neq \mathbf{H}^\dagger$ simultaneously both space reflection and time reversal invariant; in other words $[\mathbf{H}, (\mathcal{P}\mathcal{T})] = 0$. If in addition to its $\mathcal{P}\mathcal{T}$ -symmetricity, all the eigenstates of $\mathbf{H}$ are also simultaneously eigenfunction of the anti-linear symmetry $\mathcal{P}\mathcal{T}$ then the Hamiltonian $\mathbf{H}$ has real eigenvalues [4]. Since $\mathbf{H}$ is diagonalizable, and its eigenstates span a Hilbert space $\mathscr{H}$ of vectors φ_n, we can write

$$\mathbf{H}|\varphi_n\rangle = E_n|\varphi_n\rangle. \tag{1.11}$$

In addition, we assume the eigenstates are also eigenstates of the anti-linear operator product $\mathcal{P}\mathcal{T}$,

$$(\mathcal{P}\mathcal{T})|\varphi_n\rangle = \lambda_n|\varphi_n\rangle. \tag{1.12}$$

Where E_n and λ_n are respectively eigenvalues of $\mathbf{H}$ and $(\mathcal{P}\mathcal{T})$ respectively. We multiply from the right both sides of (1.12) by $(\mathcal{P}\mathcal{T})$,

$$(\mathcal{P}\mathcal{T})\mathbf{H}|\varphi_n\rangle = (\mathcal{P}\mathcal{T})E_n|\varphi_n\rangle \tag{1.13}$$

Combining the properties (1.9) and (1.12), equation (1.13) becomes

$$(\mathcal{PT})\mathbf{H}|\varphi_n\rangle = E_n^*(\mathcal{PT})|\varphi_n\rangle \quad (1.14)$$
$$(\mathcal{PT})\mathbf{H}|\varphi_n\rangle = E_n^*\lambda_n|\varphi_n\rangle \quad (1.15)$$

Similarly:

$$\begin{aligned}\mathbf{H}(\mathcal{PT})|\varphi_n\rangle &= \mathbf{H}\lambda_n|\varphi_n\rangle_n \\ &= E_n\lambda_n|\varphi_n\rangle \quad (1.16)\end{aligned}$$

Since $\mathbf{H}$ has an unbroken $\mathcal{PT}$ -symmetry, equations (1.15) and (1.16) are the same and so

$$E_n\lambda_n = E_n^*\lambda_n \quad (1.17)$$
$$E_n = E_n^* \quad (1.18)$$

We conclude that the quasi-Hermitian $\mathbf{H}$ has a real spectrum.

The extension to a much larger class of Hilbert spaces. In fact, we know that there is correlation between the properties of a linear operator and the Hilbert space where it acts. Since the similarity transformation $\mathbf{S}$ connects the quasi-Hermitian Hamiltonian $\mathbf{H}$ to a Hermitian Hamiltonian $\tilde{\mathbf{H}}$; $\mathbf{S}$ must also be the connection between the complex Hilbert space on which $\mathbf{H}$ acts and the real Hilbert space on which $\tilde{\mathbf{H}}$ acts.

$$\tilde{\mathbf{H}} = \mathbf{SHS}^{-1} \quad (1.19)$$

one can write

$$\begin{aligned}\tilde{\mathbf{H}} &= \tilde{\mathbf{H}}^\dagger \\ \mathbf{SHS}^{-1} &= \left(\mathbf{S}^\dagger\right)^{-1}\mathbf{H}^\dagger\mathbf{S}^\dagger \\ \mathbf{S}^\dagger\mathbf{SH} &= \mathbf{H}^\dagger\mathbf{S}^\dagger\mathbf{S}. \quad (1.20)\end{aligned}$$

Defining the operator $\mathbf{T}$ as

$$\mathbf{T} = \mathbf{S}^\dagger\mathbf{S} \quad (1.21)$$

it follows that

$$\mathbf{TH} = \mathbf{H}^\dagger\mathbf{T}. \quad (1.22)$$

We have come to the point where having a quasi-Hermitian Hamiltonian allows us to define a new positive definite real normed Hilbert space. As we know the concept of hermiticity is always defined with respect to an inner product. We are in a position to generalize the concept of hermiticity by allowing it to be defined positive definite real inner product. In other words a linear operator $\mathbf{H}$, fulfilling;

$$\mathbf{H}^{\dagger} = \mathbf{T}\mathbf{H}\mathbf{T}^{-1} \tag{1.23}$$

is Hermitian with respect to the T-inner product:

$$\langle \varphi_j | \mathbf{T}\mathbf{H} | \psi_k \rangle = \langle \varphi_j | \mathbf{H}^{\dagger}\mathbf{T} | \psi_k \rangle. \tag{1.24}$$

Where $|\psi_k\rangle$ are the right hand eigenstates and $|\varphi_j\rangle$ are the left hand eigenstates.

The similarity transformation (1.23) shows that even though the non-Hermiticity complicates the nature of connection between the left hand basis and the right hand basis, the metric operator $\mathbf{T}$ establish another connection between the two basis. In other words the basis of the Hilbert spaces generated by $\mathbf{H}$ and the on generated by $\mathbf{H}^{\dagger}$ are similar with respect to the metric $\mathbf{T}$.

CHAPTER 2
HERMITIZATION OF A NON-HERMITIAN OPERATOR

In the first chapter we have seen that under certain assumptions, we can extend quantum mechanics to the use of non-Hermitian operators. In fact, we have observed that if a Quasi-Hermitian operator is also $\mathcal{PT}$-symmetric with an unbroken $\mathcal{PT}$-symmetry, we can construct a similarity transformation connecting the basis generated by such a Quasi-Hermitian operator on the Hilbert space on which it acts to the basis generated by a Hermitian observable on the Hilbert space where it acts. Such a construction is also defining a metric based inner product with respect to which the Quasi-Hermitian observable is Hermitian.

In this chapter we will construct a positive definite metric for the non-Hermitian quadratic Hamiltonian[25] using the connection between the non-Hermitian Hamiltonian $\mathbf{H}$ and the Hermitian Hamiltonian $\tilde{\mathbf{H}}$. We will present the aspects related to the existence of such a metric for this particular choice, the connection between the three basis, (the right hand $|\varphi\rangle_R$ basis generated by H, the left hand basis $|\varphi\rangle_L$ basis generated by $\mathbf{H}^\dagger$, and the basis $|\varphi\rangle\rangle$ generated by the corresponding Hermitian Hamiltonian $\tilde{\mathbf{H}}$) and present the metric based inner product as an extension of the standard inner product.

2.1 Local and global form of general Bogoliubov transformations

In the last section of the first chapter we have seen that for a $\mathcal{PT}$ - symmetric Hamiltonian $\mathbf{H}$ there exists a metric $\mathbf{T}$ hermitizing $\mathbf{H}$ with respect the T *-inner product*. Such a metric is associated with the similarity transformation $\mathbf{S}$, from which there exists a Hermitian Hamiltonian $\tilde{\mathbf{H}}$ given by

$$\tilde{\mathbf{H}} = \mathbf{SHS}^{-1}. \tag{2.1}$$

As an illustration we consider a single mode of a bosonic field with frequency ω whose the Hamiltonian $\mathbf{H}$ is given by

$$\mathbf{H} = \omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right) + \alpha\mathbf{a}^2 + \beta\mathbf{a}^{\dagger 2}. \tag{2.2}$$

Here $\mathbf{a}^\dagger$ and $\mathbf{a}$ are creation and annihilation operators of the mode. These are related to $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ by:

$$\begin{cases} \mathbf{a} = \sqrt{\frac{\omega}{2}}\,\hat{\mathbf{x}} + \frac{i}{\sqrt{2\omega}}\hat{\mathbf{p}} \\ \\ \mathbf{a}^\dagger = \sqrt{\frac{\omega}{2}}\,\hat{\mathbf{x}} - \frac{i}{\sqrt{2\omega}}\hat{\mathbf{p}}, \end{cases} \tag{2.3}$$

where $\hat{\mathbf{p}} = -i\frac{d}{dx}$ in position space, and $\hat{\mathbf{x}} = i\frac{d}{dp}$ in momentum space. We recall that $\left[\hat{\mathbf{x}}, \hat{\mathbf{p}}\right] = i$. The first term is the energy of the free field (we use $\hbar = 1$). The second and the third terms describe two-bosons interacting process where α and β are the strength of interactions between two bosons respectively annihilated and created simultaneously[18] page 249. In addition α and β are products of amplitude and coupling constant due to the nonlinear susceptibility of the medium[17] page 1055.

In practice the non-Hermitian Hamiltonian (2.2) can be observed in Bose-Einstein Condensation; when a Bose gas is confined in a magnetic field with a component perpendicular to the pins[16]. The resulting many-body Hamiltonian with pairwise interaction is the non-Hermitian boson Hubbard model. When non-Hermitian boson Hubbard model Hamiltonian is considered for a diluted Bose gas it gives the non-Hermitian quadratic Hamiltonian. The same non-Hermitian Hamiltonian can also be observed in Non-Linear quantum optics using an anisotropic medium such that the intensity of the incident light where the annihilation occurs differs from the intensity of the light where the creation occurs. This phenomenon is used in signal transmission as a way to reduce the noise due to quantum fluctuations. In the literature we found the Hermitian version called '*parametric down-conversion*'[17] page 1054, [21] page 248. In Bose-Einstein Condensation; when the condensate described with a second quantized Hermitian Hamiltonian (Boson Hubbard Model or Richardson Model) is used to describe a dilute condensate at low temperature; in literature it is called '*Belieav coupling between quasi-particles*'[22], [23]. And in the quantum theory of the laser, there is a very similar process appearing in a single mode two-photon laser in which an atom in the excited state $|a\rangle$ makes a transition to the lower level $|b\rangle$ by emitting two photons via a virtual level. All these illustrations are easily observed when considering a Simple Harmonic Oscillator in the presence of a quadratic field which displaces and amplifies the wave packets. That is exactly the effect introduced in the Simple Harmonic Oscillator in the Hamiltonian (2.2), by the two-boson terms $\mathbf{a}^{\dagger 2}$ and $\mathbf{a}^2$.

On the other hand, the creation and annihilation operators $\mathbf{a}^\dagger$ and $\mathbf{a}$ are used to describe a bosonic field with n degrees of freedom; In introducing the Fock space through their action on a standard Hilbert space. This process is the second quantization of the bosonic field. It associates each mode of the field with a Simple Harmonic Oscillator; an association of each oscillator with n bosons. This shows that a set of Harmonic Oscillators is dynamically equivalent to a many-particle Bose gas. Consequently the variables attached to those modes behave like those of the quantum Harmonic Oscillators. And the quantum process can be understood through its classical representation; the Simple Harmonic Oscillator. This is a trivial case to which has been added the quadratic terms $\alpha\mathbf{a}^2 + \beta\mathbf{a}^{\dagger 2}$ which describes a mixture of the environment surrounding the process and the couplings between the bosons.

Since $\mathcal{P}\mathbf{a}\mathcal{P}^{-1} = -\mathbf{a}$ and $\mathcal{T}\mathbf{a}\mathcal{T}^{-1} = \mathbf{a}$ (similarly for $\mathbf{a}^\dagger$), the non-Hermitian Hamiltonian (2.2) is $\mathcal{PT}$ - symmetric and diagonalizable. Therefore let us consider the similarity transformation $\mathbf{S}$ such that

$$\mathbf{S} = e^{\mathbf{A}} \tag{2.4}$$

$$\mathbf{A} = \epsilon\,\mathbf{a}^\dagger\mathbf{a} + \eta\,\mathbf{a}^2 + \eta^*\,\mathbf{a}^{\dagger 2}, \tag{2.5}$$

where η is a complex parameter and η^* its complex conjugate, and ϵ is a real parameter [11]. It follows that (2.2) can explicitly be transformed as follows

$$\begin{aligned}\tilde{\mathbf{H}} &= \mathbf{SHS}^{-1}\\ &= \omega\left(\left[e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}}\right]\left[e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}}\right] + \frac{1}{2}\right)\\ &\quad + \alpha\left[e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}}\right]\left[e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}}\right] + \beta\left[e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}}\right]\left[e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}}\right]\end{aligned} \tag{2.6}$$

Using the Baker-Campbell-Hausdorff theorem, the terms $\left[e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}}\right]$ and $\left[e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}}\right]$ become

$$\begin{aligned}e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}} &= \left[\cosh\sqrt{\epsilon^2 - 4|\eta|^2} - \frac{\epsilon}{\sqrt{\epsilon^2 - 4|\eta|^2}}\sinh\sqrt{\epsilon^2 - 4|\eta|^2}\right]\mathbf{a}\\ &\quad - \frac{2\eta^*}{\sqrt{\epsilon^2 - 4|\eta|^2}}\sinh\sqrt{\epsilon^2 - 4|\eta|^2}\,\mathbf{a}^\dagger\end{aligned} \tag{2.7}$$

$$\begin{aligned}e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}} &= \frac{2\eta}{\sqrt{\epsilon^2 - 4|\eta|^2}}\sinh\sqrt{\epsilon^2 - 4|\eta|^2}\,\mathbf{a}\\ &\quad + \left[\cosh\sqrt{\epsilon^2 - 4|\eta|^2} + \frac{\epsilon}{\sqrt{\epsilon^2 - 4|\eta|^2}}\sinh\sqrt{\epsilon^2 - 4|\eta|^2}\right]\mathbf{a}^\dagger\end{aligned} \tag{2.8}$$

[11]This choice of the parameters is motivated by the requirement that $\mathbf{S}$ must be Hermitian with respect to the standard inner product on the Hilbert space $\mathscr{H}$

We define $\theta = \sqrt{\epsilon^2 - 4|\eta|^2}$

$$e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}} = \left[\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right]\mathbf{a} - \frac{2\eta^*}{\theta}\sinh\theta\mathbf{a}^\dagger \tag{2.9}$$

$$e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}} = \frac{2\eta}{\theta}\sinh\theta\mathbf{a} + \left[\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right]\mathbf{a}^\dagger \tag{2.10}$$

We substitute (2.9) and (2.10) to obtain

$$\begin{aligned}\tilde{\mathbf{H}} = &\left[\omega\left(1 - \frac{8|\eta|^2}{\theta^2}\sinh^2\theta\right) - 4\alpha\frac{\eta^*}{\theta}\sinh\theta\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)\right.\\ &\left.+4\beta\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right)\right]\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)\\ &+\left[2\omega\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right) + \alpha\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)^2 + 4\beta\frac{\eta^2}{\theta^2}\sinh^2\theta\right]\mathbf{a}^2\\ &+\left[-2\omega\frac{\eta^*}{\theta}\sinh\theta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right) + 4\alpha\frac{\eta^{*2}}{\theta^2}\sinh^2\theta + \beta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right)^2\right]\mathbf{a}^{\dagger 2}\end{aligned} \tag{2.11}$$

We define:

$$\begin{aligned}F_{\epsilon\eta}(\alpha,\beta,\omega) &= \omega\left(1 - \frac{8|\eta|^2}{\theta^2}\sinh^2\theta\right) - 4\alpha\frac{\eta^*}{\theta}\sinh\theta\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)\\ &\quad+4\beta\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right),\end{aligned} \tag{2.12}$$

$$G_{\epsilon\eta}(\alpha,\beta,\omega) = 2\omega\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right) + \alpha\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)^2 + 4\beta\frac{\eta^2}{\theta^2}\sinh^2\theta, \tag{2.13}$$

$$H_{\epsilon\eta}(\alpha,\beta,\omega) = -2\omega\frac{\eta^*}{\theta}\sinh\theta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right) + 4\alpha\frac{\eta^{*2}}{\theta^2}\sinh^2\theta + \beta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right)^2. \tag{2.14}$$

Substituting this into (2.11) gives

$$\tilde{\mathbf{H}} = F_{\epsilon\eta}(\alpha,\beta,\omega)\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right) + G_{\epsilon\eta}(\alpha,\beta,\omega)\mathbf{a}^2 + H_{\epsilon\eta}(\alpha,\beta,\omega)\mathbf{a}^{\dagger 2}. \tag{2.15}$$

For $\tilde{\mathbf{H}}$ to be Hermitian we require the coefficient of $\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}$ is real, and the coefficients of $\mathbf{a}^2$ and $\mathbf{a}^{\dagger 2}$ are complex conjugate of one another. Using these two requirements, we have:

$$F_{\epsilon\eta}(\alpha,\beta,\omega) = F^*_{\epsilon\eta}(\alpha,\beta,\omega) \tag{2.16}$$

$$G_{\epsilon\eta}(\alpha,\beta,\omega) = H^*_{\epsilon\eta}(\alpha,\beta,\omega) \tag{2.17}$$

From the equation (2.16), it follows that η is real. Inserting this into (2.17) implies that

$$\frac{1}{\theta}\tanh 2\theta = \frac{\alpha-\beta}{(\alpha+\beta)\epsilon - 2\omega\eta}. \tag{2.18}$$

On another hand $\epsilon^2 - 4\eta^2 \geq 0$ implies that $-1 \leq 2\frac{\eta}{\epsilon} \leq 1$, and consequently which holds that

$$\frac{e^{2\theta} - e^{-2\theta}}{e^{2\theta} + e^{-2\theta}} = \frac{(\alpha - \beta)\theta}{(\alpha + \beta)\epsilon - 2\omega\,\eta} \tag{2.19}$$

from which follows,

$$e^{\theta} = \left[\frac{(\alpha+\beta)\epsilon + (\alpha-\beta)\theta - 2\omega\,\eta}{(\alpha+\beta)\epsilon - (\alpha-\beta)\theta - 2\omega\,\eta}\right]^{\frac{1}{4}}. \tag{2.20}$$

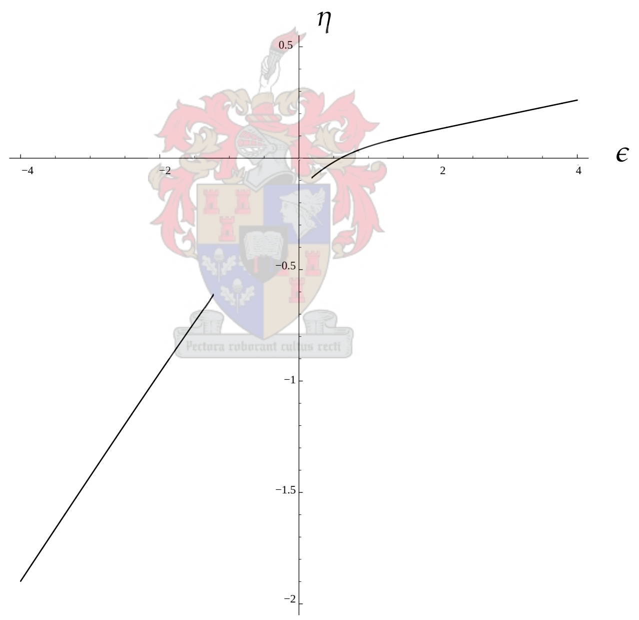

Figure 2.1: The locus of pairs (ϵ, η) for a valid metric T

The figure represents the locus of pairs (ϵ, η) for which the metric $\mathbf{T}$ defines a inner product on the Hilbert space $\mathscr{H}$ that hermitizes the non-Hermitian quadratic Hamiltonian (2.2). We also notice that there is a empty region in the plan $\epsilon 0 \eta$ where even though $-1 \leq \frac{2\eta}{\epsilon} \leq 1$ we can not find the real pairs (ϵ, η) for the metric $\mathbf{T}$. Such an interval can be identified for the asymptotic cases where for the Hermitian limit $\epsilon \longrightarrow 0$ it completely vanishes while for big enough it open wider and tend to a limiting size when $\epsilon \longrightarrow \infty$ and η small enough.

Finally the similarity transformation $\mathbf{S}$

$$\mathbf{S} = e^{\epsilon \mathbf{a}^{\dagger}\mathbf{a} + \eta(\mathbf{a}^2 + \mathbf{a}^{\dagger 2})}, \tag{2.21}$$

maps the non-Hermitian quadratic Hamiltonian (2.2) into $\tilde{\mathbf{H}}$ given by

$$\tilde{\mathbf{H}} = F_{\epsilon\eta}(\alpha, \beta, \omega)\left(\mathbf{a}^{\dagger}\mathbf{a} + \frac{1}{2}\right) + G_{\epsilon\eta}(\alpha, \beta, \omega)\left(\mathbf{a}^2 + \mathbf{a}^{\dagger 2}\right), \tag{2.22}$$

2.2 The effect of the metric on the hermitization

Let's consider the non-Hermitian quadratic Hamiltonian $\mathbf{H}$ and its hermite conjugate $\mathbf{H}^{\dagger}$. The similarity transformation $\mathbf{S}$ transforms $\mathbf{H}$ as

$$\tilde{\mathbf{H}} = \mathbf{S}\mathbf{H}\mathbf{S}^{-1} \tag{2.23}$$

$$\tilde{\mathbf{H}}^{\dagger} = \mathbf{S}\mathbf{H}^{\dagger}\mathbf{S}^{-1}. \tag{2.24}$$

In fact $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}^{\dagger}$ implies that there exists a Hilbert space $\mathscr{H}$ with an inner product such that the domains of $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{H}}^{\dagger}$ fulfil $\mathscr{D}(\tilde{\mathbf{H}}) = \mathscr{D}(\tilde{\mathbf{H}}^{\dagger}) = \mathscr{H}$. In other words, for all $\psi_j \in \mathscr{D}(\tilde{\mathbf{H}})$ and all $\phi_k \in \mathscr{D}(\tilde{\mathbf{H}}^{\dagger})$:

$$\langle \phi_j | \left(\tilde{\mathbf{H}} | \psi_k \rangle \right) = \left(\langle \phi_j | \tilde{\mathbf{H}}^{\dagger} \right) | \psi_k \rangle. \tag{2.25}$$

Also given that $\tilde{\mathbf{H}}$ is Hermitian, there exists an orthonormal basis $|n_R\rangle$ such that

$$\tilde{\mathbf{H}}|n\rangle\rangle = E_n |n\rangle\rangle, \tag{2.26}$$

and

$$\langle\langle m | n \rangle\rangle = \delta_{m,n}. \tag{2.27}$$

Using the equation (2.23) in (2.26), we can derive the following relation

$$\begin{aligned} \mathbf{S}\mathbf{H}\mathbf{S}^{-1}|n\rangle\rangle &= E_n|n\rangle\rangle \\ \mathbf{H}\mathbf{S}^{-1}|n\rangle\rangle &= E_n \mathbf{S}^{-1}|n\rangle\rangle \qquad (2.28) \\ \mathbf{H}|n\rangle\rangle &= E_n|n\rangle\rangle \qquad (2.29) \end{aligned}$$

It follows that the right hand basis [12] $|n\rangle\rangle$ can be directly deduced from the basis $|n\rangle\rangle$ [13] of its Hermitian counterpart $\tilde{\mathbf{H}}$ through the similarity transformation $\mathbf{S}$.

$$
\begin{aligned}
|n\rangle_R &= \mathbf{S}^{-1}|n\rangle\rangle && (2.30)\\
|n\rangle\rangle &= \mathbf{S}|n\rangle_R && (2.31)
\end{aligned}
$$

Taking the transpose of (2.23) we have

$$\langle\langle\langle m|\tilde{\mathbf{H}}^{\dagger} = \langle\langle\langle m|E_m \quad (2.32)$$

Since $\tilde{\mathbf{H}}^{\dagger} = \tilde{\mathbf{H}}$, we can use $\langle\langle m|$ as $\langle\langle\langle m|$.

$$
\begin{aligned}
\langle\langle m|\mathbf{S}\mathbf{H}^{\dagger}\mathbf{S}^{-1} &= \langle\langle m|E_m\\
\langle\langle m|\mathbf{S}\mathbf{H}^{\dagger} &= \langle\langle m|\mathbf{S}E_m\\
{}_L\langle m|\mathbf{H}^{\dagger} &= {}_L\langle m|E_m && (2.33)
\end{aligned}
$$

As for the right-hand basis, the left-hand basis $\langle\langle\langle m|$ of the non-Hermitian quadratic Hamiltonian $\mathbf{H}^{\dagger}$ is directly deduced from the basis $\langle\langle m|$ through the similarity transformation $\mathbf{S}$.

$$
\begin{aligned}
{}_L\langle m| &= \langle\langle m|\mathbf{S} && (2.34)\\
|m\rangle_L &= \mathbf{S}^{\dagger}|n\rangle\rangle && (2.35)\\
\langle\langle m| &= {}_L\langle m|\mathbf{S}^{-1} && (2.36)
\end{aligned}
$$

We can see how the separation right-left appears in the definitions of $|n\rangle_R$ and ${}_L\langle m|$. This is due to the non-unitarity of the similarity transformation $\mathbf{S}$.

Combining (1.21) and (2.31), we obtain

$$
\begin{aligned}
\mathbf{T}|n\rangle\rangle &= \mathbf{S}^{\dagger}\mathbf{S}\mathbf{S}^{-1}|n\rangle\rangle\\
&= \mathbf{S}^{\dagger}|n\rangle\rangle\\
&= |m\rangle_L && (2.37)
\end{aligned}
$$

[12] The basis generated on the Hilbert space $\mathscr{H}$ by the non-Hermitian quadratic Hamiltonian $\mathbf{H}$

[13] The hermiticity removes the separation right-left

We substitute (2.36) and (2.32) in the orthonormalization relation (2.28), we have

$$\begin{aligned} \langle\langle m|n\rangle\rangle &= {}_L\langle m|\mathbf{S}^{-1}\mathbf{S}|n\rangle_R \\ &= {}_L\langle m|n\rangle_R \\ &= {}_R\langle m|\mathbf{T}|n\rangle_R \end{aligned} \tag{2.38}$$

This shows that the natural substitution of the inner product $\langle\langle m|n\rangle\rangle$ should be the metric definite inner product

$${}_R\langle m|\mathbf{T}|n\rangle_R = \delta_{m,n}. \tag{2.39}$$

Therefore the metric $\mathbf{T}$ represents a linear mapping $\mathbf{T} : \mathscr{H} \longrightarrow \mathscr{H}$ such that[1]

C1 $\mathscr{D}(\mathbf{T}) = \mathscr{H}$

C2 $\mathbf{T}$ is Hermitian with respect to the normal inner product over the Hilbert space $\mathscr{H}$

C3 ${}_R\langle m|\mathbf{T}|n\rangle_R > 0; \ \ \forall\, |n\rangle_R \in \mathscr{H}$ and $|n\rangle_R \neq 0$

C4 $\mathbf{T}$ is bounded

C5 $\mathbf{TH} = \mathbf{H}^\dagger\mathbf{T}$

In conclusion, the metric $\mathbf{T}$ appears to be an exchange operator since it maps a right hand state into a left-hand state. In this case the exchange operator is Hermitian and the similarity transformation performs the change of basis from $|n\rangle_R$ to $|n\rangle_L$. The metric $\mathbf{T}$ is

$$\mathbf{T} = \mathbf{S}^\dagger\mathbf{S} = e^{2\epsilon\mathbf{a}^\dagger\mathbf{a}+2\eta(\mathbf{a}^2+\mathbf{a}^{\dagger 2})}. \tag{2.40}$$

2.3 Illustration

At the level of the non-Hermitian Quadratic Hamiltonian we have on one side; the Hamiltonian (2.2) with its spectrum $E_n = \left(n+\frac{1}{2}\right)\sqrt{\omega^2-4\alpha\beta}$ and its set of eigenfunctions $|n\rangle_R = \mathbf{S}^{-1}|n\rangle\rangle$, and on another side $\mathbf{H}^\dagger = \omega\left(\mathbf{a}^\dagger\mathbf{a}+\frac{1}{2}\right)+\beta\mathbf{a}^2+\alpha\mathbf{a}^{\dagger 2}$ with the same spectrum $E_m = \left(m+\frac{1}{2}\right)\sqrt{\omega^2-4\alpha\beta}$ and the set of its eigenstates $|m\rangle_L = \mathbf{S}^\dagger|m\rangle\rangle$. Since the states $|m\rangle\rangle$ are eigenstates of the Hermitian quadratic Hamiltonian (2.22) they can be orthonormalized such that $\langle\langle m|n\rangle\rangle = \delta_{m,n}$. Considering the ${}_R\langle m|n\rangle_R$ (or the ${}_L\langle m|n\rangle_L$) inner product leads to a very long string of evaluations where the convergence of the series is not guaranteed. in fact;

$$\begin{aligned} {}_R\langle m|n\rangle_R &= \langle\langle m|\mathbf{S}^{-2}|n\rangle\rangle \\ &= \sum_{k=0}^{\infty}\frac{(-2\epsilon)^k}{k!}\langle\langle m|\left(\mathbf{a}^\dagger\mathbf{a}+\frac{\eta}{\epsilon}\left(\mathbf{a}^2+\mathbf{a}^{\dagger 2}\right)\right)^k|n\rangle\rangle \end{aligned} \tag{2.41}$$

In general, the eigenstates $|n\rangle\rangle$ can be written as a linear combination of the eigenstates $|r\rangle$ of the particle number operator $\hat{\mathbf{n}} = \mathbf{a}^\dagger\mathbf{a}$. Therefore we can write

$$|n\rangle\rangle = \sum_{r=0}^{\infty} \lambda_r |r\rangle \tag{2.42}$$

Which in (2.40) gives

$${}_R\langle m|n\rangle_R = \sum_{k,r,s=0}^{\infty} \frac{(-2\epsilon)^k}{k!} \lambda_r^* \lambda_s \langle r| \left(\mathbf{a}^\dagger\mathbf{a} + \frac{\eta}{\epsilon}\left(\mathbf{a}^2 + \mathbf{a}^{\dagger 2}\right)\right)^k |s\rangle \tag{2.43}$$

We observe that the matrix in this form is not absolutely real or diagonal. This shows that the inner product for the $\mathcal{PT}$-symmetric Quantum mechanics should not be taken in the traditional way but rather it must be substituted by the $\mathcal{T}$ -inner product.

CHAPTER 3
DIAGONALIZATION: EIGEN-ENERGIES AND EIGENSTATES

In the previous chapter we have found a family of canonical linear transformation on the Hilbert space $\mathscr{H}$ that hermitizes the non-Hermitian quadratic Hamiltonian. These transformations constitute a change of basis from the basis $|n\rangle_R$ of $\mathbf{H}$ to the basis $|n_R\rangle$ of $\tilde{\mathbf{H}}$. This change of the basis allows for the substitution of the old non-Hermitian quadratic Hamiltonian $\mathbf{H}$ with the new Hermitian quadratic Hamiltonian $\tilde{\mathbf{H}}$, which is easier to use for diagonalization.

We need to diagonalize $\tilde{\mathbf{H}}$, in order to study its spectrum and eigenstates. This can be obtained by performing a second similarity transformation $\mathbf{B}$ that is well known in its local form as a Bogoliubov transformation. This transformation was for the first time introduced by Bogoliubov [24] in the study of a dilute Bose gas, where a Hermitian quadratic Hamiltonian very similar to $\tilde{\mathbf{H}}$, was treated.

3.1 The Bogoliubov transformation

Let's consider a unitary operator $\mathbf{B}$ on the Hilbert space $\mathscr{H}$

$$\mathbf{B} = e^{\mathbf{G}} \tag{3.1}$$

$$\mathbf{G} = i\zeta\left(\mathbf{a}^2 + \mathbf{a}^{\dagger 2}\right), \tag{3.2}$$

where ζ is real.

This is a transformation from the old bosonic operators, $\mathbf{a}^\dagger$ and $\mathbf{a}$, to a description in terms of new bosonic operators, $\mathbf{b}^\dagger$ and $\mathbf{b}$ called *quasi-particles* creation and annihilation operators. These new operators are given by

$$\begin{cases} \mathbf{b} = \mathbf{B}\mathbf{a}\mathbf{B}^\dagger = \cosh 2\zeta\, \mathbf{a} + \sinh 2\zeta\, \mathbf{a}^\dagger \\ \\ \mathbf{b}^\dagger = \mathbf{B}\mathbf{a}^\dagger\mathbf{B}^\dagger = \sinh 2\zeta\, \mathbf{a} + \cosh 2\zeta\, \mathbf{a}^\dagger, \end{cases} \tag{3.3}$$

where the pair $\mathbf{b}^\dagger$ and $\mathbf{b}$ fulfil the canonical commutation relation $\left[\mathbf{b}, \mathbf{b}^\dagger\right] = 1$. It follows that

$$\begin{aligned} \bar{\mathbf{H}} = \mathbf{B}\,\tilde{\mathbf{H}}\,\mathbf{B}^\dagger \;=\; & \left[F_{\epsilon\eta}(\alpha,\beta,\omega)\cosh 4\zeta - 2G_{\epsilon\eta}(\alpha,\beta,\omega)\sinh 4\zeta\right]\left(\mathbf{b}^\dagger\mathbf{b} + \frac{1}{2}\right) \\ & + \left[-\frac{1}{2}F_{\epsilon\eta}(\alpha,\beta,\omega)\sinh 4\zeta + G_{\epsilon\eta}(\alpha,\beta,\omega)\cosh 4\zeta\right]\left(\mathbf{b}^2 + \mathbf{b}^{\dagger 2}\right) \end{aligned} \tag{3.4}$$

For $\bar{\mathbf{H}}$ to be diagonal we require that:

$$-\frac{1}{2}F_{\epsilon\eta}(\alpha,\beta,\omega)\sinh 4\zeta + G_{\epsilon\eta}(\alpha,\beta,\omega)\cosh 4\zeta = 0. \tag{3.5}$$

It follows that

$$\begin{aligned}\tanh 4\zeta &= \frac{2G_{\epsilon\eta}(\alpha,\beta,\omega)}{F_{\epsilon\eta}(\alpha,\beta,\omega)} \\ &= \frac{2\sqrt{\alpha\beta}}{\omega}.\end{aligned} \tag{3.6}$$

This fixes uniquely the parameter ζ and consequently the Bogoliubov transformation $\mathbf{B}$ such that

$$\zeta = \ln\left[\frac{\omega + 2\sqrt{\alpha\beta}}{\omega - 2\sqrt{\alpha\beta}}\right]^{\frac{1}{8}} \tag{3.7}$$

$$\mathbf{B} = \left(\frac{\omega + 2\sqrt{\alpha\beta}}{\omega - 2\sqrt{\alpha\beta}}\right)^{\frac{i}{8}(\mathbf{a}^2+\mathbf{a}^{\dagger 2})} \tag{3.8}$$

The diagonal Hamiltonian is therefore

$$\bar{\mathbf{H}} = \left(\mathbf{b}^\dagger\mathbf{b} + \frac{1}{2}\right)\sqrt{\omega^2 - 4\alpha\beta} \tag{3.9}$$

which implies that

$$\bar{\mathbf{H}}|n_b\rangle = E_n|n_b\rangle \tag{3.10}$$

with

$$E_n = \left(n + \frac{1}{2}\right)\sqrt{\omega^2 - 4\alpha\beta} \tag{3.11}$$

$$\langle m_b|n_b\rangle = \delta_{m,n} \tag{3.12}$$

$$|n_b\rangle = \frac{\mathbf{b}^{\dagger n}}{\sqrt{n!}}|0_b\rangle \tag{3.13}$$

$$\mathbf{b}|0_b\rangle = 0 \tag{3.14}$$

Using the eigenvalue equation (3.10) as follows,

$$\begin{aligned}\bar{\mathbf{H}}|n_b\rangle &= E_n|n_b\rangle \\ \mathbf{B}\,\tilde{\mathbf{H}}\,\mathbf{B}^\dagger|n_b\rangle &= E_n|n_b\rangle \\ \tilde{\mathbf{H}}\,\mathbf{B}^\dagger|n_b\rangle &= E_n\mathbf{B}^\dagger|n_b\rangle\end{aligned} \tag{3.15}$$

We deduce successively the basis $|n\rangle\rangle$, $|n\rangle_R$ and $|n\rangle_L$ from the eigenstates $|n_b\rangle$

$$\begin{aligned} |n\rangle\rangle &= \mathbf{B}^\dagger|n_b\rangle \\ |n\rangle_R &= \mathbf{S}^{-1}\mathbf{B}^\dagger|n_b\rangle \\ |m\rangle_L &= \mathbf{S}^\dagger\mathbf{B}^\dagger|m_b\rangle \end{aligned} \tag{3.16}$$

Thus the diagonalization of the non-Hermitian quadratic Hamiltonian (2.2) is completed

$$\mathbf{H}|n\rangle_R = E_n|n\rangle_R, \tag{3.17}$$

where the eigenvalues are

$$E_n = \left(n+\frac{1}{2}\right)\Omega \tag{3.18}$$

$$\Omega = \sqrt{\omega^2-4\alpha\beta}, \tag{3.19}$$

and the eigenstates are

$$|n\rangle_R = \mathbf{S}^{-1}\mathbf{B}^\dagger|n_b\rangle \tag{3.20}$$

$$= \exp\left[-\epsilon\mathbf{a}^\dagger\mathbf{a}-\eta(\mathbf{a}^2+\mathbf{a}^{\dagger 2})\right]\left(\frac{\omega-2\sqrt{\alpha\beta}}{\omega+2\sqrt{\alpha\beta}}\right)^{\frac{i}{8}(\mathbf{a}^2+\mathbf{a}^{\dagger 2})}|n_b\rangle \tag{3.21}$$

Once again, we can check the orthonormality of the basis $|n\rangle_R$ with respect to the T-inner product

$$\begin{aligned} {}_R\langle m|\mathbf{T}|n\rangle_R &= {}_R\langle m|n\rangle_L \\ &= \langle m_b|\mathbf{B}\left(\mathbf{S}^\dagger\right)^{-1}\mathbf{S}^\dagger\mathbf{B}^\dagger|n_b\rangle \\ &= \langle m_b|n_b\rangle \\ &= \delta_{m,n} \end{aligned} \tag{3.22}$$

3.2 Swanson's diagonalization

When Swanson studied the non-Hermitian Quadratic Hamiltonian (2.2) [25], a quite interesting portion of the discussion in the study of non-Hermitian $\mathcal{PT}$-symmetric Quantum mechanics has been focused on the study of (2.2) in a number of papers [26],[28]. The study of the non-Hermitian quadratic Hamiltonian (2.2) has set up two different methods. Here we are going to connect these two approaches. The Swanson's one step transformation is based on the non-unitary canonical transformation that maps the old bosonic operators, $\mathbf{a}^\dagger$ and $\mathbf{a}$, to two new creation and destruction bosonic operators, $\mathbf{c}$ and $\mathbf{d}$, with $\mathbf{d}^\dagger \neq \mathbf{c}$ and $\mathbf{c}^\dagger \neq \mathbf{d}$:

$$\begin{cases} \mathbf{c} = g_1\mathbf{a}^\dagger - g_3\mathbf{a} \\ \mathbf{d} = g_4\mathbf{a} - g_2\mathbf{a}^\dagger. \end{cases} \tag{3.23}$$

From the canonical commutation relation $\left[\mathbf{d}, \mathbf{c}\right] = \left[\mathbf{a}, \mathbf{a}^\dagger\right] = 1$, it follows that

$$g_1g_4 - g_2g_3 = 1 \tag{3.24}$$

Substituting (3.23) in the non-Hermitian quadratic Hamiltonian (2.2), and requiring that the quadratic terms in $\mathbf{c}^2$ and $\mathbf{d}^2$ vanish, one can derive the following relations between the g_i 's:

$$\frac{g_3}{g_1} = \frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\beta}, \tag{3.25}$$

$$\frac{g_2}{g_4} = \frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\alpha}, \tag{3.26}$$

$$g_1g_2 = -\frac{\beta}{\sqrt{\omega^2 - 4\alpha\beta}}, \tag{3.27}$$

$$g_2g_3 = \frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}, \tag{3.28}$$

$$g_3g_4 = -\frac{\alpha}{\sqrt{\omega^2 - 4\alpha\beta}} \tag{3.29}$$

$$g_1g_4 + g_2g_3 = \frac{\omega}{\sqrt{\omega^2 - 4\alpha\beta}}. \tag{3.30}$$

And the transformed Hamiltonian becomes

$$\mathbf{H}_s = \left(\mathbf{cd} + \frac{1}{2}\right)\sqrt{\omega^2 - 4\alpha\beta}. \tag{3.31}$$

There is a number of questions that need to be cleared up in Swanson's approach:

First: the Hamiltonian $\mathbf{H}_s$ can be diagonal if and only if $\mathbf{cd} = \mathbf{d}^\dagger\mathbf{c}^\dagger$. This requirement is met since the combinations of the g_i 's as it appears in the above relations (from (3.25) to (3.30)) are real:

Second: the definition of the right-hand states $|\tilde{n}\rangle$ must be considered solely at the level of the diagonal Hamiltonian $\mathbf{H}_s$ and not be extended to the non-Hermitian quadratic Hamiltonian $\mathbf{H}$. Otherwise we need to define a similarity transformation whose the local form is given by (3.23) connecting the basis $|\tilde{n}\rangle$ to the basis generated by the non-Hermitian quadratic Hamiltonian on the Hilbert space. In other words this similarity transformation connects the basis $|\tilde{n}\rangle$ and $|\bar{m}\rangle$ generated by $\mathbf{H}$ and $\mathbf{H}^{\dagger}$ respectively to the diagonal Hamiltonian [14] It follows that, we have on one side

$$\mathbf{H}_s|\tilde{n}\rangle = E_n|\tilde{n}\rangle \tag{3.32}$$
$$E_n = \left(n+\frac{1}{2}\right)\sqrt{\omega^2-4\alpha\beta} \tag{3.33}$$
$$|\tilde{n}\rangle = \frac{\mathbf{c}^n}{\sqrt{n!}}|0_d\rangle, \tag{3.34}$$

and on the other side

$$\mathbf{H}_s^{\dagger}|\bar{m}\rangle = E_m|\bar{m}\rangle \tag{3.35}$$
$$E_m = \left(m+\frac{1}{2}\right)\sqrt{\omega^2-4\alpha\beta} \tag{3.36}$$
$$|\bar{m}\rangle = \frac{\mathbf{d}^m}{\sqrt{m!}}|0_c\rangle, \tag{3.37}$$

Swanson introduces the metric $\mathbf{U}$ (cfr [25] page 591 equation (39))

$$\mathbf{U} = \exp\left[\frac{1}{2}\left(\frac{g_3^*}{g_1^*}-\frac{g_2}{g_4}\right)\mathbf{a}^{\dagger 2}\right]\exp\left(\frac{1}{2}w\mathbf{d}^2\right)\exp\left(\mathbf{cd}\ln z\right) \tag{3.38}$$

such that

$$\mathbf{Uc} = \mathbf{d}^{\dagger}\mathbf{U}, \tag{3.39}$$
$$\mathbf{Ud} = \mathbf{c}^{\dagger}\mathbf{U}, \tag{3.40}$$

and

$$\mathbf{U}|\tilde{n}\rangle = |\bar{n}\rangle \tag{3.41}$$

From what we already know, we derive the similarity transformation connecting $\mathbf{H}_s$ to $\mathbf{H}$:

$$\mathbf{H}_s = \mathbf{U}^{\frac{1}{2}}\mathbf{H}\mathbf{U}^{-\frac{1}{2}} \tag{3.42}$$

We can then deduce the right-hand states will be $\mathbf{U}^{-\frac{1}{2}}|\tilde{n}\rangle$ and the left-hand states will be $\mathbf{U}^{\frac{1}{2}}|\bar{m}\rangle$. This result conserves Swanson orthonormalization at all levels of the transformations

[14] Otherwise we would have to consider that $|\tilde{n}\rangle = |\bar{m}\rangle$ and $\mathbf{d}^{\dagger} = \mathbf{c}$.

(cfr [25]). Therefore

$$\mathbf{H}\mathbf{U}^{-\frac{1}{2}}|\tilde{n}\rangle = E_n \mathbf{U}^{-\frac{1}{2}}|\tilde{n}\rangle, \tag{3.43}$$

where $E_n = \left(n + \frac{1}{2}\right)\Omega$ as in (3.18). At this point, there must exist a connection between the two pictures: As it appears, this connection is easily detected since similarly to (3.20)must be a change of basis from one description to another.

$$|n\rangle_R = \mathbf{U}^{-\frac{1}{2}}|\tilde{n}\rangle \tag{3.44}$$

Let's construct a *projection-like operator* $\mathbf{\Pi}$ mapping the states $|n\rangle_L$ on the states $|\bar{m}\rangle$. Such a projection-like operator can be constructed as follows

$$\mathbf{\Pi} = \sum_{n=0}^{\infty} |\bar{n}\rangle_R \langle n| \ , \tag{3.45}$$

and

$$\mathbf{\Pi}^{-1} = \sum_{m=0}^{\infty} |m\rangle_L \langle \tilde{m}| \ . \tag{3.46}$$

This implies that

$$\begin{aligned} \mathbf{\Pi}|n\rangle_L &= \sum_{k=0}^{\infty} |\bar{k}\rangle_R \langle k|n\rangle_L \\ &= \sum_{k=0}^{\infty} |\bar{k}\rangle \delta_{k,n} \\ &= |\bar{n}\rangle \end{aligned} \tag{3.47}$$

Using this result, we can construct

$$\begin{aligned} \sum_{n=0}^{\infty} |\bar{n}\rangle\langle\bar{n}| &= \sum_{n=0}^{\infty} \mathbf{\Pi}|n_L\rangle\langle n_L|\mathbf{\Pi}^{\dagger} \\ &= \mathbf{\Pi}\mathbf{T}\mathbf{\Pi}^{\dagger}, \end{aligned} \tag{3.48}$$

and therefore

$$\mathbf{U} = \mathbf{\Pi}\mathbf{T}\mathbf{\Pi}^{\dagger}. \tag{3.49}$$

Since $\mathbf{U}$ can also be written in this fashion $\sum_{n=0}^{\infty} |\bar{n}\rangle\langle\bar{n}|$.
The result (3.49) shows that the Swanson approach (one step process) and the two step approach are equivalent up to the projection-like operator $\mathbf{\Pi}$.

CHAPTER 4
METRIC AND THE UNIQUENESS

A quantum mechanical description of any given system supposes that we have chosen: A Hilbert space on which the physical states of the system will be represented, and a set of observables relevant for the complete description of the physical system. In standard quantum mechanics this scenario is based on the hermiticity of operators and the Hilbert space. Suppose that the system we need to study is described by a $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian. In this case, the concept observables will be associated with Hermitian operators with respect to the T -inner product [12]. In other words, in addition to the choice of the Hilbert space $\mathscr{H}$, we need a metric that will fix whether or not an operator is an observable.

As it appears in the above lines, the metric plays a major role in the $\mathcal{PT}$-symmetric quantum theory framework. Indeed every time we intend to describe a physical system represented by a non-Hermitian Hamiltonian we need the Hilbert space representing the physical states and a metric $\mathbf{T}$ that hermitizes the non-Hermitian Hamiltonian with respect to the inner product. The problem is that this metric is not unique. The non-uniqueness is due to the fact that there exist various ways of constructing a metric that hermitizes the $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian. Such a constructed metric contains a certain degree of freedom. The study of the non-Hermitian quadratic Hamiltonian (2.2) reveals two possibles approaches; the two step approach where the metric $\mathbf{T}$ is only determined up to the parameter η, and the one step approach where the Swanson metric $\mathbf{U}$ is defined up to one of the coefficient g_i. We need to enforce additional constraints in order to fix uniquely the inner product and thereby realise a unique $\mathcal{PT}$-symmetric non-Hermitian quantum mechanics framework.

4.1 Observables in non-Hermitian quantum theory framework

In fact, in $\mathcal{PT}$-symmetric quantum theory we need to require an operator to be Hermitian with respect to the T -inner product before it becomes a non-Hermitian $\mathcal{PT}$-symmetric quantum theory observable. Therefore, a particular choice of observables is strongly correlated to a metric. In other words, when we chose a particular set of operators, we must require them to be Hermitian with respect to the newly defined T -inner product. This requirement is an extension of the hermiticity of the standard quantum mechanics to the set of non-Hermitian operators. In doing this we have constructed a set of non-Hermitian operator associated with the metric.

In other words an operator $\mathbf{A}_k$ is a non-Hermitian $\mathcal{PT}$-symmetric quantum theory observable if it fulfils

$$\mathbf{A}_k\mathbf{T} = \mathbf{T}\mathbf{A}_k^{\dagger}. \tag{4.1}$$

We consider two examples with the standard quantum theory position operator $\hat{\mathbf{x}}$ and momentum operator $\hat{\mathbf{p}}$. From the requirement (4.1), $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are $\mathcal{PT}$-symmetric quantum theory observables only under the requirements

$$\hat{\mathbf{x}}\mathbf{T} = \mathbf{T}\hat{\mathbf{x}}, \tag{4.2}$$

$$\hat{\mathbf{p}}\mathbf{T} = \mathbf{T}\hat{\mathbf{p}}. \tag{4.3}$$

We recall that $\mathbf{T} = \mathbf{S}^{\dagger}\mathbf{S}$ (1.21). We first multiply from the left each side of both equations (4.2) and (4.3) by $\mathbf{S}^{-1}$, and then next from the right each side of both equations (4.2) and (4.3) by $\mathbf{S}^{-1}$, we obtain in each case

$$\mathbf{S}^{-1}\hat{\mathbf{x}}\mathbf{S} = \mathbf{S}\hat{\mathbf{x}}\mathbf{S}^{-1}, \tag{4.4}$$

$$\mathbf{S}^{-1}\hat{\mathbf{p}}\mathbf{S} = \mathbf{S}\hat{\mathbf{p}}\mathbf{S}^{-1}. \tag{4.5}$$

Translated in its local form on one hand the equation (4.4) is fulfilled for

$$\frac{\epsilon - 2\eta}{\theta}\sinh\theta = 0, \tag{4.6}$$

$$\eta = \frac{\epsilon}{2} \tag{4.7}$$

which implies that the standard position operator $\hat{\mathbf{x}} = \frac{1}{\sqrt{2\omega}}\left(\mathbf{a} + \mathbf{a}^{\dagger}\right)$ can be a $\mathcal{PT}$-symmetric quantum theory observable only for $\eta = \frac{\epsilon}{2}$ in the metric $\mathbf{S}$.on the other hand the equation (4.5) is fulfilled for

$$\frac{\epsilon + 2\eta}{\theta}\sinh\theta = 0, \tag{4.8}$$

$$\eta = -\frac{\epsilon}{2} \tag{4.9}$$

which implies that the standard momentum operator $\hat{\mathbf{p}} = i\sqrt{\frac{\omega}{2}}\left(\mathbf{a}^{\dagger} - \mathbf{a}\right)$ can be a $\mathcal{PT}$-symmetric quantum theory observable only for $\eta = -\frac{\epsilon}{2}$ in the metric $\mathbf{S}$.

Consequently, these two standard quantum theory observables appear not be $\mathcal{PT}$-symmetric quantum theory observables for the same value of the free parameter η.

4.2 Non-uniqueness in expectation values

When a non-Hermitian operator can be hermitized for a wide range of the parameter η, its expectation value depends on the parameter η as well. Such a situation implies that there is a large number of different expectation values mathematically valid. Physically, it is a violation to the uniqueness in expectation values guaranteed by the standard quantum theory. As an example, we consider the transition occurring in the physical system represented by the non-Hermitian quadratic Hamiltonian (2.2) when it takes a transition from a state $|m\rangle$ with m bosons, to its eigenstate $|n\rangle_R$ with n bosons. The transition matrix element is given by;

$$w_{mn} = {}_R\langle n|\mathbf{T}e^{-i\mathbf{H}\tau}|m\rangle, \tag{4.10}$$

where τ is the duration of the transition. Since $\mathbf{T}|n\rangle_R = |n\rangle_L$, we obtain

$$\begin{aligned} w_{mn} &= {}_L\langle n|e^{-i\mathbf{H}\tau}|m\rangle \\ &= \langle n_b|\mathbf{BS}e^{-i\mathbf{H}\tau}|m\rangle \\ &= \langle n_b|\mathbf{BS}e^{-i\mathbf{H}\tau}\mathbf{S}^{-1}\mathbf{S}|m\rangle \\ &= \langle n_b|\mathbf{B}e^{-i\tilde{\mathbf{H}}\tau}\mathbf{S}|m\rangle \\ &= e^{-iE_n\tau}\langle n_b|\mathbf{BS}|m\rangle \end{aligned} \tag{4.11}$$

The matrix element $\langle n_b|\mathbf{BS}|m\rangle$ depends on the similarity transformation $\mathbf{S} = e^{\epsilon\mathbf{a}^\dagger\mathbf{a}+\eta(\mathbf{a}^2+\mathbf{a}^{\dagger 2})}$. For each value of the parameter η, we will have different value of $\langle n_b|\mathbf{BS}|m\rangle$. Here we consider the three limiting cases $\eta = 0$, $\eta = \frac{\epsilon}{2}$, and $\eta = \frac{\epsilon}{2}$. In each case, we evaluate the probability of transition $|w_{mn}|^2$.

We first consider the case where the parameter $\eta = 0$, in which case $\epsilon = \frac{1}{4}\log\left[\frac{\alpha}{\beta}\right]$. The similarity transformation becomes $\mathbf{S} = \left(\frac{\alpha}{\beta}\right)^{\hat{\mathbf{n}}/4}$. which in (4.11) gives

$$w_{mn} = e^{-iE_n\tau}\langle n_b|\mathbf{B}\left(\frac{\alpha}{\beta}\right)^{\frac{\hat{\mathbf{n}}}{4}}|m\rangle. \tag{4.12}$$

Which gives

$$\begin{aligned} w_{mn} &= \left[\frac{\alpha}{\beta}\right]^{\frac{m}{4}} e^{-iE_n\tau}\langle n_b|\mathbf{B}|m\rangle \\ &= \left[\frac{\alpha}{\beta}\right]^{\frac{m}{4}} e^{-iE_n t}\lambda^{(0)}_{mn}, \end{aligned} \tag{4.13}$$

where

$$\lambda^{(0)}_{mn} = \langle n_b|\mathbf{B}|m\rangle. \tag{4.14}$$

The details on the evaluation of $\lambda^{(0)}_{mn}$ are presented in Appendix C.

$$\lambda^{(0)}_{mn} = \begin{cases} 0 \quad For\ m+n\ odd \\ \\ \Big[\frac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)^{m+n+1}}\Big]^{\frac{1}{2}} \quad For\ m+n\ even \end{cases} \tag{4.15}$$

We substitute $\lambda^{(0)}_{mn}$ from (4.15) in the expression of w_{mn} (4.13), and obtain

$$w_{mn} = \begin{cases} 0 \quad For \quad m+n \quad odd \\ \\ \Big[\frac{\alpha}{\beta}\Big]^{\frac{m}{4}}\Big[\frac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)^{m+n+1}}\Big]^{\frac{1}{2}} e^{-iE_n t} \quad For \quad m+n \quad even \end{cases} \tag{4.16}$$

The transition takes place only from the states $|m\rangle$ to the states $|n\rangle_R$ fulfilling the condition $m+n \quad even$, with the probability

$$|w_{mn}|^2 = 2\Big[\frac{\alpha}{\beta}\Big]^{\frac{m}{2}} \frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)^{m+n+1}} \tag{4.17}$$

Similarly to the transition from the states $|m\rangle$ to the states $|n\rangle_R$, the reverse transition taking from the eigenstates $|n\rangle_R$ to the states $|m\rangle$; has the probability

$$|w_{nm}|^2 = 2\Big[\frac{\alpha}{\beta}\Big]^{\frac{m}{2}} \frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)^{m+n+1}} \tag{4.18}$$

More details are presented in Appendix C. We can see that $|w_{nm}|^2 = |w_{mn}|^2$; which means that the probability in both ways transitions is conserved !

Next we consider the case where $\eta = \frac{\epsilon}{2}$, the similarity transformation is $\mathbf{S} = e^{\epsilon\hat{\mathbf{x}}^2}$. Which substituted in (4.11) gives

$$w_{mn} = e^{-iE_n\tau}\langle n_b|\mathbf{B}e^{\epsilon\hat{\mathbf{x}}^2}|m\rangle. \tag{4.19}$$

The transition takes place only from the states $|m\rangle$ to the states $|n\rangle_R$ fulfilling the condition

$m+n$ *even*, with the probability

$$|w_{mn}|^2 = 2\frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}+\frac{\alpha-\beta}{\omega-\alpha-\beta}\right)^{m+n+1}}, \tag{4.20}$$

while the reversed transition taking place from the eigenstates $|n\rangle_R$ to the states $|m\rangle$. occurs also only for $m+n$ *even*, with the probability

$$|w_{nm}|^2 = 2\frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}+\frac{\alpha-\beta}{\omega-\alpha-\beta}\right)^{m+n+1}}. \tag{4.21}$$

In this case as well the probability in both ways transitions is conserved $|w_{nm}|^2 = |w_{mn}|^2$ .More details are presented in Appendix C.

Next we consider the case where $\eta = -\frac{\epsilon}{2}$, the similarity transformation is $\mathbf{S} = e^{\epsilon\hat{\mathbf{p}}^2}$. Which substituted in (4.11) gives

$$w_{mn} = e^{-iE_n\tau}\langle n_b|\mathbf{B}e^{\epsilon\hat{\mathbf{x}}^2}|m\rangle. \tag{4.22}$$

The transition takes place only from the states $|m\rangle$ to the states $|n\rangle_R$ fulfilling the condition $m+n$ *even*, with the probability

$$|w_{mn}|^2 = 2\frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}+\frac{\alpha-\beta}{\omega+\alpha+\beta}\right)^{m+n+1}}, \tag{4.23}$$

while the reversed transition taking place from the eigenstates $|n\rangle_R$ to the states $|m\rangle$. occurs also only for $m+n$ *even*, with the probability

$$|w_{nm}|^2 = 2\frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}+\frac{\alpha-\beta}{\omega+\alpha+\beta}\right)^{m+n+1}}. \tag{4.24}$$

In this case as for the two previous cases, the probability in both ways transitions is conserved $|w_{nm}|^2 = |w_{mn}|^2$ .More details are presented in Appendix C.

We can also notice that at the Hermitian limit $\alpha \longrightarrow \beta$ all the values of the free parameter η give the same probability of transition.

$$|w_{mn}|^2 = 2\frac{\omega^m\left(\omega^2-4\alpha^2\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha^2}\right)^{m+n+1}}. \tag{4.25}$$

The evaluation of the probability of transition $|w_{nm}|^2$ for the non-Hermitian quadratic Hamiltonian shows that it depends on the parameter η of the metric $\mathbf{T}$. In choosing the three cases $\eta = 0$, $\eta = \frac{\epsilon}{2}$, and $\eta = -\frac{\epsilon}{2}$ reveal, different values for the probability; even though in each one of these cases the probability is conserved in both ways transition.

This non-uniqueness in the probability of transition is an illustration of a deeper problem encounter when dealing with free parameters in a metric in $\mathcal{PT}$-symmetric quantum theory. As presented here all values of the free parameter give valid outcomes, but which value of the free parameter is the most relevant? What are the physical reasons behind a particular choice of such a choice?

4.3 Discussion on uniqueness

The correlation between metric and the representation of observables, the suitability of metric with respect to some observable, and the non-uniqueness in expectation values shows that the construction of metric must be connected with a number of requirements. These requirements are to be directly deduced from the structure of the system under study and the set of observables relevant for a complete description of the physical system.

The question related to the uniqueness of the metric all related problems where addressed in a general study of Quasi-Hermitian Hamiltonian. [1](Which includes the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian) This original work showed that relatively to the set of non-Hermitian observables to be used in the complete study of a "*quasi-Hermitian system*[15]". There exists an irreducible set of observables on $\mathscr{H}$ with respect to which the metric can be uniquely defined (up to a global normalization factor). As it appears today, this statement is a central theorem on the uniqueness of the metric. In a way it appears that the irreducible set should be a subset extracted on the set of observables required for a complete description of the non-Hermitian $\mathcal{PT}$-symmetric quantum system.

Let us consider a set $\mathscr{A}$ of all observables $\mathbf{A}_i$ needed for the complete description of a non-Hermitian $\mathcal{PT}$-symmetric quantum system.

Definition 4.3.1 *A subset $\mathscr{A}_K$ of $\mathscr{A}$ is an irreducible set if it can not be written as the union of two proper nonempty closed subsets of the $\mathscr{A}$*

Theorem 4.3.1 *Let us consider an linear operator* $\mathbf{T} : \mathscr{H} \longrightarrow \mathscr{H}$.

If $\mathbf{T}$ *satisfies:*

P1 $\mathscr{D}(\mathbf{T}) = \mathscr{H}$

[15]When using quasi-Hermitian system, we understand a system represented by a quasi-Hermitian observable or Hamiltonian. The same remark is true for non-Hermitian $\mathcal{PT}$-symmetric system.

P2 $\mathbf{T}$ *is Hermitian with respect to the normal inner product over the Hilbert space* $\mathscr{H}$

P3 ${}_R\langle m|\mathbf{T}|n\rangle_R > 0; \quad \forall\, |n\rangle_R \in \mathscr{H}$ *and* $|n\rangle_R \neq 0$

P4 $\mathbf{T}$ *is bounded*

P5 $\mathbf{T}\mathbf{A}_i = \mathbf{A}_i^{\dagger}\mathbf{T}$,

Then $\mathbf{T}$ *is a one to one mapping [1] (up to a global normalization constant).*

Proof 4.3.1 *Let us assume that the metric* $\mathbf{T}$ *satisfies all the five properties: P1, P2, P3, P4, and P5 on the Hilbert space* $\mathscr{H}$*. First we need to show that* $\mathbf{T}^{-1}$ *exists and* $\mathscr{D}(\mathbf{T})$ *is the whole Hilbert space* $\mathscr{H}$*. To show that* $\mathbf{T}^{-1}$ *exists, we must show that* $\mathbf{T}$ *is a one to one mapping.*

Suppose that $\mathbf{T}$ *is not a one to one mapping, then there must exist a ket* $|n_0\rangle_R$ *non-zero such that* $\mathbf{T}|n_0\rangle_R = 0$ *. Hence* ${}_R\langle n_0|\mathbf{T}|n_0\rangle_R = 0$ *. This contradicts the condition P3 the 'positive definiteness' of* $\mathbf{T}$*. Suppose, furthermore, that* $\mathscr{D}(\mathbf{T}^{-1}) = \mathscr{R}(\mathbf{T}) \subsetneqq \mathscr{H}$ *; (* $\mathscr{R}(\mathbf{T})$ *is the range or image of* $\mathbf{T}$ *). There we can find a non-zero ket* $|n_0\rangle_R$ *in* $\left(\mathscr{R}(\mathbf{T})\right)^{\perp}$ *(The set of vectors in the Hilbert space* $\mathscr{H}$ *which are orthogonal to* $\mathscr{R}(\mathbf{T})$ *.), and* ${}_R\langle n_0|\mathbf{T}|n_0\rangle_R = 0$ *, which again contradicts the positive definiteness of* $\mathbf{T}$*. Therefore* $\mathbf{T}$ *must be a one to one mapping and consequently* $\mathbf{T}^{-1}$ *exists.*

Next we show that $\mathbf{T}^{-1}$ *is bounded. From the property P1* $\mathbf{T}$ *is bounded and* $\mathscr{D}(\mathbf{T}) = \mathscr{H}$*, which implies that* $\mathbf{T}$ *and* $\mathbf{T}^{-1}$ *are closed [29]. Combining this with the closed graph theorem[29] page166, we deduce that* $\mathbf{T}^{-1}$ *is bounded. Earlier, we have established in the equation (1.21) that the metric* $\mathbf{T} = \mathbf{S}^{\dagger}\mathbf{S}$*. This result (1.21) combined to the boundedness of the metric* $\mathbf{T}$ *implies that* $\mathbf{S}$ *and* $\mathbf{S}^{-1}$ *are also bounded.*

Now we need to show that there exist an Hilbert space $\mathscr{H}_T$ *endowed with the* T *-inner product. Let us define a norm* $(\rho_n)_T = {}_R\langle n|\mathbf{T}|n\rangle_R^{\frac{1}{2}}$*. Suppose that* $\left\{\rho_k\right\}_{k=1}^{\infty}$ *is a Cauchy sequence with respect to the norm* $(\rho_n)_T$ *(With* $\rho_k = {}_R\langle n|n\rangle_R^{\frac{1}{2}}$ *). We consider a vector* $|n_{ij}\rangle_R = |n_j\rangle_R - |n_i\rangle_R$*. Given that* $\mathbf{S}$ *and* $\mathbf{S}^{-1}$ *are bounded, and considering the norm* $\rho_{ij} = {}_R\langle n_{ij}|n_{ij}\rangle_R^{\frac{1}{2}}$*, we can make the following consideration:*

$|n_{ij}\rangle_R = \mathbf{S}^{-1}\mathbf{S}|n_{ij}\rangle_R^{\frac{1}{2}} = \mathbf{S}^{-1}\left(\mathbf{S}|n_{ij}\rangle_R\right)$, $\rho_{ij} \leq \|\mathbf{S}^{-1}\|_R\langle n_{ij}|\mathbf{S}^{\dagger}\mathbf{S}|n_{ij}\rangle_R^{\frac{1}{2}} = \|\mathbf{S}^{-1}\|_R\langle n_{ij}|\mathbf{T}|n_{ij}\rangle_R^{\frac{1}{2}}$. *Hence* $\left\{\rho_k\right\}_{k=1}^{\infty}$ *is a Cauchy sequence in the norm* $\rho_n = {}_R\langle n|n\rangle_R^{\frac{1}{2}}$*. Since* $\mathscr{H}$ *is complete with respect to the norm* ρ_n*, there exists a ket* $|n\rangle_R$ *in* $\mathscr{H}$ *such that any given ket* $|n_{nk}\rangle_R = |n_k\rangle_R - |n_n\rangle_R$ *has its norm* $\rho_{nk} \longrightarrow 0$ *as* $k \longrightarrow \infty$ *. Then the boundedness of* $\mathbf{S}$ *yields* ${}_R\langle n_{nk}|\mathbf{T}|n_{nk}\rangle_R^{\frac{1}{2}} \longrightarrow 0$ *as* $k \longrightarrow \infty$ *. Hence the sequence* $\left\{\rho_k\right\}_{k=1}^{\infty}$ *converges to* $|n\rangle_R \in \mathscr{H}$ *in the norm* $(\rho_n)_T$*. We conclude that* $\mathscr{H}_T$*, endowed with the T-inner product forms a Hilbert space* $\mathscr{H}_T$*.*

Now we need to prove that the metric $\mathbf{T}$ *is unique on* $\mathscr{H}$ *if, and only if, the set* $\mathscr{A}_K$ *of observables fulfilling the property P5 is irreducible on the Hilbert space* $\mathscr{H}$*. First we prove that if the irreducibility holds,* $\mathbf{T}$ *is unique. For this propose we need the following proposition[31] page 47:*

Proposition *A set of Hermitian bounded operators on a Hilbert space* $\mathscr{H}$ *is irreducible if, and only if, every bounded operator that commutes with every element of this set is proportional to the identity.*

Proof *By assumption the non-Hermitian observables* $\mathbf{A}_i$ *are bounded on the Hilbert space* $\mathscr{H}$*. Then they are bounded on* $\mathscr{H}_T$*, and it follows* ${}_R\langle n|\mathbf{T}\mathbf{A}_i|n\rangle_R^{\frac{1}{2}} = \|\mathbf{S}\mathbf{A}_i\mathbf{S}^{-1}\|_R\, {}_R\langle n|\mathbf{T}|n\rangle_R^{\frac{1}{2}}$*. Furthermore they are Hermitian with respect to the T-inner product, and by assumption they are irreducible. Let us assume that there exists an operator* $\mathbf{T}'$ *satisfying all the properties of* $\mathbf{T}$*. Then it follows from P5 that* $\left[\mathbf{T}^{-1}\mathbf{T}',\, \mathbf{A}_i\right] = 0$ *, for all* i *.*

Clearly from the previous proof, $\mathbf{T}^{-1}\mathbf{T}'$ *is bounded. Hence* $\mathbf{T}^{-1}\mathbf{T}'$ *is at the simultaneously proportional to the identity and to* $\mathbf{T}'$ *, therefore, it is proportional to* $\mathbf{T}$*. To prove the converse we simply have to prove that if the set* $\mathscr{A}_K$ *is reducible,* $\mathbf{T}$ *is not unique. This can be easily accomplished by the following construction:*

Suppose that the set $\mathscr{A}_K$ *is reducible on* $\mathscr{H}$*, and therefore also on* $\mathscr{H}_T$*. Then we can find the last proper subspace of* $\mathscr{H}_T$ *which is left invariant under the set* $\mathscr{A}_K$*, and since this observables are Hermitian on* $\mathscr{H}_T$*, they also leave the orthogonal complement of this subspace in* $\mathscr{H}_T$ *invariant. Let* $\mathbf{P}$ *denotes the Hermitian projection operator on* $\mathscr{H}_T$ *that project onto this subspace and* $\mathbf{Q}$ *denotes the Hermitian projection operator on* $\mathscr{H}_T$ *that project onto its orthogonal complement in* $\mathscr{H}_T$*. Since* $\mathbf{P}$ *and* $\mathbf{Q}$ *are Hermitian on* $\mathscr{H}_T$ *it follows that*

$$\mathbf{T}\mathbf{P} = \mathbf{P}^{\dagger}\mathbf{T} \tag{4.26}$$

$$\mathbf{T}\mathbf{Q} = \mathbf{Q}^{\dagger}\mathbf{T} \tag{4.27}$$

One also deduce that $\mathbf{Q}^{\dagger}\mathbf{P}^{\dagger} = \mathbf{P}^{\dagger}\mathbf{Q}^{\dagger}$, $\left(\mathbf{P}^{\dagger}\right)^2 = \mathbf{P}^{\dagger}$, $\left(\mathbf{Q}^{\dagger}\right)^2 = \mathbf{Q}^{\dagger}$*. Let us construct an operator* $\mathbf{V} = \mu\mathbf{P} + \nu\mathbf{Q}$ *with* μ *and* ν *positive real numbers. We maintain that* $\mathbf{T}' = \mathbf{T}\mathbf{V}$*, which is in general not proportional to* $\mathbf{T}$*, and satisfies all the properties p1, p2, p3, p4, and p5. Clearly the property P1 holds. Following (4.26) and (4.27)* $\mathbf{T}'$ *is Hermitian. It also grant the positive definiteness since* ${}_R\langle\varphi|\mathbf{T}'|\varphi\rangle_R = \mu\, {}_R\langle\varphi|\mathbf{T}\mathbf{P}|\varphi\rangle_R + \nu\, {}_R\langle\varphi|\mathbf{T}\mathbf{Q}|\varphi\rangle_R \geq 0$ *, for all non-zero vectors* $|\varphi\rangle_R$*.* $\mathbf{T}'$ *is clearly bounded since* $\mathbf{V}$ *is bounded. Finally, it follows from the invariance of* $\mathbf{P}\mathscr{H}$ *and* $\mathbf{Q}\mathscr{H}$ *under the set* $\mathscr{A}_K$ *that* $\left[\mathbf{A}_k,\, \mathbf{V}\right] = 0$ *, for all* k *. Multiplying* $\mathbf{T}\mathbf{A}_k = \mathbf{A}_k^{\dagger}\mathbf{T}$ *from the right with* $\mathbf{V}$ *gives* $\mathbf{T}'\mathbf{A}_k = \mathbf{A}_k^{\dagger}\mathbf{T}'$ *.*

We have come to the point where the irreducible set $\mathscr{A}_K$ guarantees the uniqueness of the metric $\mathbf{T}$.

4.4 Irreducible sets and uniqueness of the metric

The section 4.1 presents three consequences of the non-uniqueness of the metric. These consequences are caused by the free parameters in the metric. We have seen that the irreducible set $\mathscr{A}_K$ provides us with observables $\mathbf{A}_k$ such that

$$\mathbf{T}\mathbf{A}_k = \mathbf{A}_k^\dagger \mathbf{T}. \tag{4.28}$$

This requirement (4.28) fixes the metric's free parameters η uniquely. It implies that the metric is therefore uniquely defined.

For illustration, let us consider the metric $\mathbf{T}$ (2.40). This metric has one free parameter η, with ϵ being the *global normalization factor* predicted in the central theorem 4.1.1. It appears that in the case of the metric $\mathbf{T}$, the irreducible set will have the cardinality of two elements: The Hamiltonian (2.2) and an additional observable. Here we should be aware of the fact that there may be more than one irreducible sets. The multiplicity of irreducible sets will conduct to different uniquely defined metric each of them valid within particular framework. Here we present the cases where we consider the metric $\mathbf{T}$ for three different irreducible sets; $\left\{\mathbf{H}, \hat{\mathbf{n}}\right\}$, $\left\{\mathbf{H}, \hat{\mathbf{x}}\right\}$, and $\left\{\mathbf{H}, \hat{\mathbf{p}}\right\}$ each of them leading to a proper $\mathcal{PT}$-symmetric quantum theory.

4.4.1 The irreducible set $\left\{\mathbf{H}, \hat{\mathbf{n}}\right\}$

Let us consider the requirement (4.28) for the particle number operator $\hat{\mathbf{n}}$

$$\mathbf{T}\hat{\mathbf{n}} = \hat{\mathbf{n}}\mathbf{T}. \tag{4.29}$$

We substitute $\mathbf{T}$ by $\mathbf{S}^\dagger\mathbf{S}$,

$$\begin{aligned} \mathbf{S}\hat{\mathbf{n}}\mathbf{S}^{-1} &= \mathbf{S}^{-1}\hat{\mathbf{n}}\mathbf{T} \\ \mathbf{S}\mathbf{a}^\dagger\mathbf{a}\mathbf{S}^{-1} &= \mathbf{S}^{-1}\mathbf{a}^\dagger\mathbf{a}\mathbf{S}. \end{aligned} \tag{4.30}$$

Expanding the left hand side and the right hand side and comparing, we obtain

$$\left(\cosh^2\theta - \frac{\epsilon^2}{\theta^2}\sinh^2\theta\right)\mathbf{a}^\dagger\mathbf{a} = \left(\cosh^2\theta - \frac{\epsilon^2}{\theta^2}\sinh^2\theta\right)\mathbf{a}^\dagger\mathbf{a}, \tag{4.31}$$

$$-4\frac{\eta^2}{\theta^2}\sinh^2\theta\,\mathbf{a}\mathbf{a}^\dagger = -4\frac{\eta^2}{\theta^2}\sinh^2\theta\,\mathbf{a}\mathbf{a}^\dagger, \tag{4.32}$$

$$2\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)\mathbf{a}^2 = -2\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right)\mathbf{a}^2, \tag{4.33}$$

$$-2\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right)\mathbf{a}^{\dagger 2} = 2\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)\mathbf{a}^{\dagger 2}. \tag{4.34}$$

Equations (4.31) and (4.32) are just trivial equalities. From equations (4.33) and (4.34) we obtain the same result

$$2\frac{\eta}{\theta}\cosh\theta = 0 \tag{4.35}$$

Which gives

$$\eta = 0 \tag{4.36}$$

Therefore the metric $\mathbf{T}$ becomes

$$\mathbf{T} = e^{2\epsilon \mathbf{a}^{\dagger}\mathbf{a}} \tag{4.37}$$

Using $\eta = 0$ in the expression (2.20) $e^{\theta} = \left[\frac{(\alpha+\beta)\epsilon+(\alpha-\beta)\theta-2\omega\,\eta}{(\alpha+\beta)\epsilon-(\alpha-\beta)\theta-2\omega\,\eta}\right]^{\frac{1}{4}}$ we obtain

$$\epsilon = \ln\left[\frac{\alpha}{\beta}\right]^{\frac{1}{4}} \tag{4.38}$$

Thus the metric $\mathbf{T}$ takes the form which was obtained in [26] by H.B. Geyer, F.G. Scholtz, and Izak Snyman

$$\mathbf{T} = \left(\frac{\alpha}{\beta}\right)^{\frac{\hat{\mathbf{n}}}{2}} \tag{4.39}$$

4.4.2 The irreducible set $\left\{\mathbf{H}, \hat{\mathbf{x}}\right\}$

Suppose that the metric $\mathbf{T}$ commutes with the position operator. The requirement (4.28), we will become the commutation relation

$$\left[\mathbf{T}, \hat{\mathbf{x}}\right] = 0. \tag{4.40}$$

It follows

$$\begin{aligned} \mathbf{T}\hat{\mathbf{x}} - \hat{\mathbf{x}}\mathbf{T} &= 0 \\ \mathbf{T}\hat{\mathbf{x}} &= \hat{\mathbf{x}}\mathbf{T} \\ \mathbf{S}^{\dagger}\mathbf{S}\hat{\mathbf{x}} &= \hat{\mathbf{x}}\mathbf{S}^{\dagger}\mathbf{S} \end{aligned} \tag{4.41}$$

We use the hermiticity of $\mathbf{S}$ and multiply (4.41) by $\mathbf{S}^{-1}$ from the left and from the right the resulting expression. We obtain

$$\mathbf{S}\hat{\mathbf{x}}\mathbf{S}^{-1} = \mathbf{S}^{-1}\hat{\mathbf{x}}\mathbf{S}. \tag{4.42}$$

We expand both sides of (4.40), and obtain

$$\begin{aligned} &\left(\cosh\theta - \frac{\epsilon - 2\eta}{\theta}\sinh\theta\right)\mathbf{a} \quad + \quad \left(\cosh\theta + \frac{\epsilon - 2\eta}{\theta}\sinh\theta\right)\mathbf{a}^\dagger \\ = &\left(\cosh\theta + \frac{\epsilon - 2\eta}{\theta}\sinh\theta\right)\mathbf{a} \quad + \quad \left(\cosh\theta - \frac{\epsilon - 2\eta}{\theta}\sinh\theta\right)\mathbf{a}^\dagger. \end{aligned} \tag{4.43}$$

In comparing the coefficients of (4.43), we find that

$$\frac{\epsilon - 2\eta}{\theta}\sinh\theta = 0 \tag{4.44}$$

Which implies that

$$\eta = \frac{\epsilon}{2} \tag{4.45}$$

Which substituted in (2.18) gives

$$\epsilon = -\frac{1}{2}\frac{\alpha - \beta}{\omega - \alpha - \beta} \tag{4.46}$$

Therefore the metric operator $\mathbf{T}$ takes the form

$$\mathbf{T} = \exp\left(-\frac{\alpha - \beta}{\omega - \alpha - \beta}\omega\,\hat{\mathbf{x}}^2\right), \tag{4.47}$$

which was predicted by Hugh Jones in [28].

4.4.3 The irreducible set $\{\mathbf{H}, \hat{\mathbf{p}}\}$

Suppose that the metric $\mathbf{T}$ hermitizes the momentum operator. The requirement (4.28), we will become the commutation relation

$$\left[\mathbf{T}, \hat{\mathbf{p}}\right] = 0. \tag{4.48}$$

It follows

$$\begin{aligned} \mathbf{T}\hat{\mathbf{p}} - \hat{\mathbf{p}}\mathbf{T} &= 0 \\ \mathbf{T}\hat{\mathbf{p}} &= \hat{\mathbf{p}}\mathbf{T} & (4.49) \\ \mathbf{S}^\dagger\mathbf{S}\hat{\mathbf{p}} &= \hat{\mathbf{p}}\mathbf{S}^\dagger\mathbf{S} & (4.50) \end{aligned}$$

We use the hermiticity of $\mathbf{S}$ and multiply (4.50) by $\mathbf{S}^{-1}$ from the left and from the right the resulting expression. We obtain

$$\mathbf{S}\hat{\mathbf{p}}\mathbf{S}^{-1} = \mathbf{S}^{-1}\hat{\mathbf{p}}\mathbf{S}. \tag{4.51}$$

We expand both sides of (4.50), and obtain

$$\begin{aligned}&\Big(\cosh\theta - \frac{\epsilon+2\eta}{\theta}\sinh\theta\Big)\mathbf{a} \;-\; \Big(\cosh\theta + \frac{\epsilon+2\eta}{\theta}\sinh\theta\Big)\mathbf{a}^\dagger\\ =&\Big(\cosh\theta + \frac{\epsilon+2\eta}{\theta}\sinh\theta\Big)\mathbf{a} \;-\; \Big(\cosh\theta - \frac{\epsilon+2\eta}{\theta}\sinh\theta\Big)\mathbf{a}^\dagger.\end{aligned} \tag{4.52}$$

In comparing the coefficients of (4.52), we find that

$$\frac{\epsilon+2\eta}{\theta}\sinh\theta = 0 \tag{4.53}$$

Which implies that

$$\eta = -\frac{\epsilon}{2} \tag{4.54}$$

Which substituted in (2.18) gives

$$\epsilon = \frac{1}{2}\frac{\alpha-\beta}{\omega+\alpha+\beta} \tag{4.55}$$

Therefore the metric operator $\mathbf{T}$ takes the form

$$\mathbf{T} = \exp\Big(\frac{1}{\omega}\frac{\alpha-\beta}{\omega+\alpha+\beta}\hat{\mathbf{p}}^2\Big), \tag{4.56}$$

which was predicted by Hugh Jones in [28].

We have already seen that there exists multiple choices of irreducible sets. The arbitrary choice we have made of three different irreducible sets: $\{\mathbf{H}, \hat{\mathbf{n}}\}$, $\{\mathbf{H}, \hat{\mathbf{x}}\}$, and $\{\mathbf{H}, \hat{\mathbf{p}}\}$ has demonstrated that $\{\mathbf{H}, \hat{\mathbf{p}}\}$ and $\{\mathbf{H}, \hat{\mathbf{x}}\}$ are the two extreme cases with all others lying between them. In other words, the fact that the metric $\mathbf{T}$ is valid for $-\frac{\epsilon}{2} \le \eta \le \frac{\epsilon}{2}$ means that at $\eta = -\frac{\epsilon}{2}$ the irreducible set corresponds to the pair $\{\mathbf{H}, \hat{\mathbf{p}}\}$, and at $\eta = \frac{\epsilon}{2}$ the irreducible set corresponds to the pair $\{\mathbf{H}, \hat{\mathbf{x}}\}$. With $\eta = 0$ corresponding to the irreducible set $\{\mathbf{H}, \hat{\mathbf{n}}\}$.

The fact that we have the parameter η lying $-\frac{\epsilon}{2} \le \eta \le \frac{\epsilon}{2}$ shows that when dealing with operators with regards to their dependencies on the position and momentum. By requiring them to be observables, the degrees of the dependence with respect to each one of the momentum and the position determine in which side of the zero the parameter η will be localized. Such an observation can help to anticipate in the understanding of the study of a physical system. The interpretation of expectation values will have additional information coming from the localization of the parameter; information related to more physical effects.

4.5 Comparison between Swanson framework and the two steps framework

We have come to the point where starting from a non uniquely defined metric $\mathbf{T}$, we have come to different metric $\mathbf{T}$ uniquely defined using the irreducible set. We have also seen that the metric $\mathbf{T}$ is the same with the Swanson's metric $\mathbf{U}$. The comparison between the two methods can be established. We start requiring the uniqueness of the metric $\mathbf{U}$ using the same method.

4.5.1 Commutation between the metric $\mathbf{U}$ and $\hat{\mathbf{n}}$

We require commutation between the Swanson's metric $\mathbf{U}$ (3.38) and the particle number operator $\hat{\mathbf{n}} = \mathbf{a}^\dagger \mathbf{a}$.

$$\mathbf{U}\hat{\mathbf{n}} = \hat{\mathbf{n}}\mathbf{U}. \tag{4.57}$$

This requirement fixes the choice of one of the g_j s and establishes all uniquely such that they are explicitly:

$$\begin{cases} g_1 = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}} \left[\frac{\omega + \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}} e^{i\varphi} \\ g_2 = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}} \left[\frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}} e^{-i\varphi} \\ g_3 = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \left[\frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}} e^{i\varphi} \\ g_4 = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \left[\frac{\omega + \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}} e^{-i\varphi} \end{cases} \tag{4.58}$$

On the other hand, we consider

$$\begin{aligned} \mathbf{U}\mathbf{a} &= \mathbf{U}(g_1\mathbf{d} + g_2\mathbf{c}) \\ &= ((|g_1|^2 - |g_2|^2)\mathbf{a}\mathbf{U}, \end{aligned} \tag{4.59}$$

which yields

$$\mathbf{a}\mathbf{U} = (|g_1|^2 - |g_2|^2)^{-1}\mathbf{U}\mathbf{a}. \tag{4.60}$$

The Hermitian conjugate gives

$$\begin{aligned} \mathbf{U}\mathbf{a}^\dagger &= (|g_1|^2 - |g_2|^2)^{-1}\mathbf{a}^\dagger\mathbf{U} \\ \mathbf{a}^\dagger\mathbf{U} &= (|g_1|^2 - |g_2|^2)\mathbf{U}\mathbf{a}^\dagger. \end{aligned} \tag{4.61}$$

Similarly

$$\mathbf{U}\mathbf{a}^\dagger = ((|g_4|^2 - |g_3|^2)\mathbf{a}^\dagger\mathbf{U} \tag{4.62}$$

which yields

$$\mathbf{a}^{\dagger}\mathbf{U} \ = \ (|g_4|^2 - |g_3|^2)^{-1}\mathbf{U}\mathbf{a}^{\dagger} \tag{4.63}$$

The result (4.63) is equivalent to (4.61). This shows that the procedure is consistent. We substitute $|g_4|^2 - |g_3|^2 = \frac{g_3^*}{g_2}$ (cfr Appendix D), we obtain

$$\mathbf{aU} \ = \ \frac{g_3^*}{g_2}\mathbf{Ua} \tag{4.64}$$

$$\mathbf{a}^{\dagger}\mathbf{U} \ = \ \frac{g_2}{g_3^*}\mathbf{U}\mathbf{a}^{\dagger} \tag{4.65}$$

The requirement (4.57) implies also that the metric $\mathbf{U} = \mathbf{U}(n)$. Therefore we ca write

$$\mathbf{aU}(n) = \mathbf{U}(n-1)\mathbf{a} \tag{4.66}$$

Combining (4.64) and (4.62), we obtain

$$\mathbf{U}(n+1) = \frac{g_3^*}{g_2}\mathbf{U}(n) \tag{4.67}$$

Solving the difference equation (4.63), we obtain

$$\mathbf{U} \ = \ \left(\frac{g_3^*}{g_2}\right)^{\hat{\mathbf{n}}} \tag{4.68}$$

$$\mathbf{U} \ = \ \left(\frac{\alpha}{\beta}\right)^{\frac{\hat{\mathbf{n}}}{2}} \tag{4.69}$$

The last expression is obtained after the substitution of (4.58) in (4.69). The result (4.69) is equivalent to the metric in [26] and what we found by requiring the metric $\mathbf{T}$ to commute with the irreducible set $\left\{\mathbf{H}, \hat{\mathbf{x}}\right\}$ (equivalent to the parameter η being zero).

4.5.2 Commutation between the metric $\mathbf{U}$ and $\hat{\mathbf{x}}$

Here, we consider that the Swanson's metric $\mathbf{U} = \mathbf{U}(x)$, where is the position. (We recall that the position operator is given by $\hat{\mathbf{x}} = \frac{1}{\sqrt{2\omega}}\left(\mathbf{a} + \mathbf{a}^{\dagger}\right)$). And we require $\hat{\mathbf{x}}$ to fulfil

$$\mathbf{U}\hat{\mathbf{x}} = \hat{\mathbf{x}}\mathbf{U} \tag{4.70}$$

Similarly to the previous case we obtain all the g_i uniquely defined

$$\begin{cases} g_1 &= \left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1-\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{\frac{1}{2}} \\ g_2 &= -\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1-\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{-\frac{1}{2}} \\ g_3 &= -\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1-\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{\frac{1}{2}} \\ g_4 &= \frac{\omega+\sqrt{\omega^2-4\alpha\beta}}{2\sqrt{\omega^2-4\alpha\beta}}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1-\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{-\frac{1}{2}} \end{cases} \tag{4.71}$$

Given that the metric $\mathbf{U} = \mathbf{U}(x)$. We can use the identity [30] page 140

$$-i\frac{d\mathbf{U}}{dx} = \left[\hat{\mathbf{p}}, \mathbf{U}(x)\right]. \tag{4.72}$$

We expand the commutator on left hand side of (4.72), and express $\hat{\mathbf{p}}$ in $\mathbf{a}$ and $\mathbf{a}^\dagger$ to obtain

$$\begin{aligned} -i\frac{d\mathbf{U}}{dx} &= \hat{\mathbf{p}}\mathbf{U} - \mathbf{U}\hat{\mathbf{p}} \\ &= \hat{\mathbf{p}}\mathbf{U} + i\sqrt{\frac{\omega}{2}}\left[(g_1 - g_3)\mathbf{c}^\dagger + (g_2 - g_4)\mathbf{d}^\dagger\right]\mathbf{U}. \end{aligned} \tag{4.73}$$

We substitute $\mathbf{a}$ and $\mathbf{a}^\dagger$ by their expressions in (2.3). Using the canonical condition $g_1 g_4 - g_2 g_3 = 1$, we obtain

$$\begin{aligned} -i\frac{d\mathbf{U}}{dx} &= \hat{\mathbf{p}}\mathbf{U} + i\left[g_1^2 - g_2^2 - 1\right]]\hat{\mathbf{x}}\mathbf{U} - \hat{\mathbf{p}}\mathbf{U} \\ -i\frac{d\mathbf{U}}{dx} &= i\left[g_1^2 - g_2^2 - 1\right]]\hat{\mathbf{x}}\mathbf{U}. \end{aligned} \tag{4.74}$$

The separation of variables gives

$$\mathbf{U}^{-1}d\mathbf{U} = \left[1 - g_1^2 + g_2^2\right]\hat{\mathbf{x}}dx \tag{4.75}$$

$$\mathbf{U} = \exp\left(\frac{1}{2}\left[1 - g_1^2 + g_2^2\right]\hat{\mathbf{x}}^2\right) \tag{4.76}$$

We substitute in (4.76) g_1 and g_2 by their values from (4.71), which gives

$$\mathbf{U} = \exp\left(-\frac{\alpha - \beta}{\omega - \alpha - \beta}\omega\hat{\mathbf{x}}^2\right) \tag{4.77}$$

In the last expression (4.77), the metric $\mathbf{U}$ is equivalent to the metric ρ in [28] page 1743 equation (11) and what we found by requiring the metric $\mathbf{T}$ to commute with the irreducible set $\{\mathbf{H}, \hat{\mathbf{x}}\}$ (equivalent to the parameter η being $\frac{\epsilon}{2}$).

4.5.3 Commutation between the metric $\mathbf{U}$ and $\hat{\mathbf{p}}$

Similarly to the commutation between $\mathbf{U}$ and $\hat{\mathbf{x}}$, the commutation $\mathbf{U}$ and $\hat{\mathbf{p}}$ leads to the uniquely determined g_i 's give by the expressions

$$\begin{cases} g_1 &= \left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1-\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{\frac{1}{2}} \\ g_2 &= -\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1-\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{-\frac{1}{2}} \\ g_3 &= -\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1-\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{\frac{1}{2}} \\ g_4 &= \frac{\omega+\sqrt{\omega^2-4\alpha\beta}}{2\sqrt{\omega^2-4\alpha\beta}}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1-\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{-\frac{1}{2}} \end{cases} \quad (4.78)$$

In addition, we consider that the Swanson's metric $\mathbf{U} = \mathbf{U}(p)$. Therefore it fits identity

$$-i\frac{d\mathbf{U}}{dp} = \left[\hat{\mathbf{x}}, \mathbf{U}(p)\right], \quad (4.79)$$

[30] page 140. Similarly to what was performed in the previous section, algebraic manipulations gives differential the equation

$$\frac{d\mathbf{U}}{dp} = (g_1^2 - g_2^2 - 1)\hat{\mathbf{p}}\mathbf{U}. \quad (4.80)$$

The separation of variables gives

$$\mathbf{U}^{-1}d\mathbf{U} = \left(g_1^2 - g_2^2 - 1\right)\hat{\mathbf{p}}dp \quad (4.81)$$

$$\mathbf{U} = \exp\left(\frac{1}{2}\left(g_1^2 + g_2^2 - 1\right)\hat{\mathbf{p}}^2\right) \quad (4.82)$$

We substitute in (4.82) g_1 and g_2 by their values from (4.78), which gives

$$\mathbf{U} = \exp\left(\frac{1}{\omega}\frac{\alpha-\beta}{\omega+\alpha+\beta}\hat{\mathbf{p}}^2\right) \quad (4.83)$$

In the last expression (4.83), the metric $\mathbf{U}$ is equivalent to what we found by requiring the metric $\mathbf{T}$ to commute with the irreducible set $\{\mathbf{H}, \hat{\mathbf{p}}\}$ (equivalent to the parameter η being $-\frac{\epsilon}{2}$).

In conclusion, we have seen that the metric can be uniquely defined, when we require it to hermitize all observables in the irreducible set. After such a process we obtain a metric that can only be used in the evaluation of observables related to the irreducible set: In other words, suppose that we chose the metric for the irreducible set $\{\mathbf{H}, \hat{\mathbf{x}}\}$, we can only evaluate the expectation value of observables Hermitian with respect to the T -inner product for $\eta = \frac{\epsilon}{2}$.

The fact that the Swanson one step approach metric $\mathbf{U}$ and the two step approach metric $\mathbf{T}$ are the same under the irreducible sets hermitization requirement (4.28) can be already enlighten by considering the asymptotic case $\theta \to 0$ combined to demanding

$$G_{\epsilon\eta}(\alpha, \beta, \omega) = 0 \tag{4.84}$$

$$H_{\epsilon\eta}(\alpha, \beta, \omega) = 0, \tag{4.85}$$

in the Hamiltonian (2.15). Combining these two requirements, we obtain

$$\frac{\eta^2}{\theta^2}\frac{\sinh^2\theta}{\cosh^2\theta - \frac{\epsilon^2}{\theta^2}\sinh^2\theta} = -\frac{1}{4}. \tag{4.86}$$

From which the following observations can be made:

When θ move toward 0 from the positive values, it appears to be not possible to fulfil the requirement (4.86) since we know that $\sinh\theta < \cosh\theta$ for $\theta < \infty$. But when θ move toward 0 from the negative values, the requirement (4.86) becomes

$$\frac{\eta^2}{\theta^2}\frac{\sin^2\theta}{\cos^2\theta - \frac{\epsilon^2}{\theta^2}\sin^2\theta} = -\frac{1}{4}, \tag{4.87}$$

and holds only for $\cos^2\theta < \frac{\epsilon^2}{\theta^2}\sin^2\theta$, which can also be translated into these inequalities

$$-\frac{\epsilon}{\theta}\sin\theta < \cos\theta < \frac{\epsilon}{\theta}\sin\theta \tag{4.88}$$

This two last equations show that the one step transformation $\mathbf{T}$ can be used to diagonalize the Hamiltonian (2.2) in one step when ϵ , and η are chosen such that they satisfy (4.88). In other words, there exist some choices of the parameters ϵ and η that enabling the metric $\mathbf{T}$ to diagonalize the non-Hermitian Hamiltonian (2.2). Looking at these particular aspects may be interesting.

4.6 Example: Quasi-Hermitian shifted oscillator

We are going to illustrate the above method with the non-Hermitian shifted quantum harmonic oscillator. We consider the $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian for the shifted oscillator

$$\mathbf{H} = \frac{1}{2}\Big(\hat{\mathbf{p}}^2 + \omega^2\hat{\mathbf{x}}^2\Big) + i\omega\hat{\mathbf{x}} \tag{4.89}$$

We substitute $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ by their respective expressions from (2.3), we obtain

$$\mathbf{H} = \omega\Big(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\Big) + i\sqrt{\frac{\omega}{2}}\Big(\mathbf{a} + \mathbf{a}^\dagger\Big) \tag{4.90}$$

We introduce a similarity transformation $\mathbf{S}$ that maps the non-Hermitian Hamiltonian $\mathbf{H}$ on a Hermitian Hamiltonian $\tilde{\mathbf{H}}$. Suppose that

$$\mathbf{S} = \exp\Big[\eta\mathbf{a} + \eta^*\mathbf{a}^\dagger\Big] \tag{4.91}$$

Using the Baker-Campbell-Haussdorf theorem, we derive

$$e^{\eta\mathbf{a}+\eta^*\mathbf{a}^\dagger}\mathbf{a}\, e^{-\eta\mathbf{a}-\eta^*\mathbf{a}^\dagger} = \mathbf{a} - \eta^* \tag{4.92}$$

$$e^{\eta\mathbf{a}+\eta^*\mathbf{a}^\dagger}\mathbf{a}^\dagger e^{-\eta\mathbf{a}-\eta^*\mathbf{a}^\dagger} = \mathbf{a}^\dagger + \eta \tag{4.93}$$

Therefore

$$\tilde{\mathbf{H}} = \mathbf{S}\mathbf{H}\mathbf{S}^{-1} \tag{4.94}$$

$$\begin{aligned}\tilde{\mathbf{H}} &= \omega\Big(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\Big) + i\sqrt{\frac{\omega}{2}}\Big(1 - i\eta\sqrt{2\omega}\Big)\mathbf{a} \\ &\quad + i\sqrt{\frac{\omega}{2}}\Big(1 + i\eta^*\sqrt{2\omega}\Big)\mathbf{a}^\dagger + i\sqrt{\frac{\omega}{2}}\Big(\eta - \eta^* + i|\eta|^2\sqrt{2\omega}\Big)\end{aligned} \tag{4.95}$$

Given that $\tilde{\mathbf{H}}$ is Hermitian, we must have

$$i\sqrt{\frac{\omega}{2}}\Big(1 - i\eta\sqrt{2\omega}\Big) = \Big[i\sqrt{\frac{\omega}{2}}\Big(1 + i\eta^*\sqrt{2\omega}\Big)\Big]^* \tag{4.96}$$

$$i\sqrt{\frac{\omega}{2}}\Big(\eta - \eta^* + i|\eta|^2\sqrt{2\omega}\Big) = \Big[i\sqrt{\frac{\omega}{2}}\Big(\eta - \eta^* + i|\eta|^2\sqrt{2\omega}\Big)\Big]^* \tag{4.97}$$

which can also be written as

$$i\sqrt{\frac{\omega}{2}}\Big(1 - i\eta\sqrt{2\omega}\Big) = -i\sqrt{\frac{\omega}{2}}\Big(1 - i\eta\sqrt{2\omega}\Big) \tag{4.98}$$

$$i\sqrt{\frac{\omega}{2}}\Big(\eta - \eta^* + i|\eta|^2\sqrt{2\omega}\Big) = -i\sqrt{\frac{\omega}{2}}\Big(-\eta + \eta^* - i|\eta|^2\sqrt{2\omega}\Big) \tag{4.99}$$

With the last requirement (4.99) being a trivial equality, the first requirement (4.98) imposes

$$\begin{aligned} 2i\sqrt{\frac{\omega}{2}}\left(1 - i\eta\sqrt{2\omega}\right) &= 0 \\ \eta &= \frac{-i}{\sqrt{2\omega}} \end{aligned} \tag{4.100}$$

We substitute η in (4.99), and obtain

$$i\sqrt{\frac{\omega}{2}}\left(\eta - \eta^{*} + i|\eta|^{2}\sqrt{2\omega}\right) = \frac{1}{2} \tag{4.101}$$

Therefore, the Hermitian Hamiltonian $\tilde{\mathbf{H}}$ becomes

$$\tilde{\mathbf{H}} = \omega\left(\mathbf{a}^{\dagger}\mathbf{a} + \frac{1}{2}\right) + \frac{1}{2}, \tag{4.102}$$

and the similarity transformation $\mathbf{S}$ becomes

$$\mathbf{S} = \exp\left[\frac{i}{\sqrt{2\omega}}\left(\mathbf{a}^{\dagger} - \mathbf{a}\right)\right] \tag{4.103}$$

Using the definition (1.21) for the metric, we obtain

$$\begin{aligned} \mathbf{T} &= \mathbf{S}^{\dagger}\mathbf{S} \\ &= \exp\left(i\sqrt{\frac{2}{\omega}}\left(\mathbf{a}^{\dagger} - \mathbf{a}\right)\right) \end{aligned} \tag{4.104}$$

Considering the momentum operator $\hat{\mathbf{p}} = i\sqrt{\frac{\omega}{2}}(\mathbf{a}^{\dagger} - \mathbf{a})$, the metric $\mathbf{T}$ becomes

$$\mathbf{T} = \exp\left(\frac{2}{\omega}\hat{\mathbf{p}}\right) \tag{4.105}$$

The hermitization of the non-Hermitian shifted oscillator leads to a uniquely defined metric $\mathbf{T}$. This metric is similar to the one used by C. M. Bender, Jun-Hua Chen and K. A Milton [5] in the construction of the operator $\mathcal{C} = \mathbf{T}\mathcal{P}$ On the other hand, $\tilde{\mathbf{H}}$ is diagonal and the spectrum of eigenvalues and eigenstates are

$$\begin{aligned} \tilde{\mathbf{H}}|n\rangle &= E_{n}|n\rangle \\ &= \omega\left(n + \frac{1}{2} + \frac{1}{2\omega}\right)|n\rangle \end{aligned} \tag{4.106}$$

Using the similarity transformation (4.94), we deduce

$$\begin{aligned} \mathbf{H}\mathbf{S}^{-1}|n\rangle &= E_n\mathbf{S}^{-1}|n\rangle \\ |n\rangle_R &= \mathbf{S}^{-1}|n\rangle \end{aligned} \tag{4.107}$$

with the eigenvalues and the right-hand eigenstates respectively given by the following expressions

$$E_n = \omega\Big(n + \frac{1}{2} + \frac{1}{2\omega}\Big) \tag{4.108}$$

$$|n\rangle_R = e^{-\frac{1}{\omega}\hat{\mathbf{p}}}|n\rangle \tag{4.109}$$

and

$$\begin{aligned} \mathbf{H}^{\dagger}\mathbf{S}|m\rangle &= E_m\mathbf{S}|m\rangle \\ |m\rangle_L &= \mathbf{S}|m\rangle \end{aligned} \tag{4.110}$$

with the eigenvalues and the left-hand eigenstates respectively given by the following expressions

$$E_m = \omega\Big(m + \frac{1}{2} + \frac{1}{2\omega}\Big) \tag{4.111}$$

$$|m\rangle_L = e^{\frac{1}{\omega}\hat{\mathbf{p}}}|m\rangle \tag{4.112}$$

Considering the three operators: the particle number operator $\hat{\mathbf{n}} = \mathbf{a}^{\dagger}\mathbf{a}$, the position operator $\hat{\mathbf{x}}$, and the momentum operator $\hat{\mathbf{p}}$, we notice that the only physical observable is the momentum operator $\hat{\mathbf{p}}$. In fact, using

$$\mathbf{T}\mathbf{a} = \Big(\mathbf{a} - i\sqrt{\frac{2}{\omega}}\Big)\mathbf{T} \tag{4.113}$$

$$\mathbf{T}\mathbf{a}^{\dagger} = \Big(\mathbf{a}^{\dagger} - i\sqrt{\frac{2}{\omega}}\Big)\mathbf{T} \tag{4.114}$$

we can easily deduce that

$$\mathbf{T}\hat{\mathbf{n}} = \left(\hat{\mathbf{n}} - 2i\hat{\mathbf{x}} - \frac{4}{\omega}\right)\mathbf{T} \tag{4.115}$$

$$\mathbf{T}\hat{\mathbf{x}} = \left(\hat{\mathbf{x}} - 2i\sqrt{\frac{2}{\omega}}\right)\mathbf{T} \tag{4.116}$$

$$\mathbf{T}\hat{\mathbf{p}} = \hat{\mathbf{p}}\mathbf{T} \tag{4.117}$$

The equations (4.115), (4.116) and (4.117) present what can be deduced from the structure of the metric (4.105). Infact, the metric based inner product in this case allows only the momentum operator in addition to the Hamiltonian to be Hermitian. But it is possible by proceding from the Hermitian side with the Hamiltonian (4.102), to identify the corresponding position and particle number operators Hermitian with respect to the metric based inner product.

CHAPTER 5
PHYSICAL ASPECTS AND SPECIFIC CASES

5.1 Physical aspects

The determination of the uniquely defined metric using irreducible sets introduces various ways of obtaining a uniquely defined metric over the Hilbert space on which are represented the states of the system given by the non-Hermitian quadratic Hamiltonian $\mathbf{H}$ (2.2). Each one of these ways describes the physical system using only a particular set of operators hermitized by the metric $\mathbf{T}$ for the particular value of the parameter η corresponding to that particular set. In other words, for a set $\{\mathbf{H}, \mathbf{A}_k\}$; where $\mathbf{A}_k$ is the additional observable of the set. There exists a value of the parameter η which hermitize the non-Hermitian quadratic Hamiltonian $\mathbf{H}$ and the observable $\mathbf{A}_k$ such that the similarity transformation $\mathbf{S} = \mathbf{T}^{\frac{1}{2}}$ maps the Hamiltonian (2.2) on the Hermitian quadratic Hamiltonian $\tilde{\mathbf{H}}$ given by (2.22)

$$\tilde{\mathbf{H}} = F_{\epsilon\eta}(\alpha,\beta,\omega)\left(\mathbf{a}^{+}\mathbf{a} + \frac{1}{2}\right) + G_{\epsilon\eta}(\alpha,\beta,\omega)\left(\mathbf{a}^{2} + \mathbf{a}^{\dagger 2}\right),$$

which has the canonical form

$$\begin{aligned}\tilde{\mathbf{H}} &= \frac{1}{2\omega}\Big(\big[F_{\epsilon\eta}(\alpha,\beta,\omega) - 2G_{\epsilon\eta}(\alpha,\beta,\omega)\big]\hat{\mathbf{p}}^{2} + \big[F_{\epsilon\eta}(\alpha,\beta,\omega) + 2G_{\epsilon\eta}(\alpha,\beta,\omega)\big]\omega^{2}\hat{\mathbf{x}}^{2}\Big) \\ &= \frac{1}{2}\big[\mu_{\epsilon\eta}(\alpha,\beta,\omega)\hat{\mathbf{p}}^{2} + \nu_{\epsilon\eta}(\alpha,\beta,\omega)\hat{\mathbf{x}}^{2}\big]. \qquad (5.1)\end{aligned}$$

where we define

$$\mu_{\epsilon\eta}(\alpha,\beta,\omega) = \frac{1}{\omega}\big[F_{\epsilon\eta}(\alpha,\beta,\omega) - 2G_{\epsilon\eta}(\alpha,\beta,\omega)\big], \qquad (5.2)$$

$$\nu_{\epsilon\eta}(\alpha,\beta,\omega) = \omega\big[F_{\epsilon\eta}(\alpha,\beta,\omega) + 2G_{\epsilon\eta}(\alpha,\beta,\omega)\big]. \qquad (5.3)$$

The Hamiltonian (5.1) presents the perfect symmetry between the position and the momentum. This property characterized the quantum oscillator. In position representation the Hamiltonian (5.1) leads to the differential equation

$$\frac{1}{2}\Big[\mu_{\epsilon\eta}(\alpha,\beta,\omega)\frac{d^{2}}{dx^{2}} + E - \nu_{\epsilon\eta}(\alpha,\beta,\omega)x^{2}\Big]\psi(x) = 0. \qquad (5.4)$$

which admits for $E_n = \left(n + \frac{1}{2}\right)\sqrt{\omega^2 - 4\alpha\beta}$, the solution

$$\psi_n(x) = \frac{(-i)^n}{\sqrt{2^n n!}} \exp\Big[-\frac{1}{2}\sqrt{\frac{\nu_{\epsilon\eta}(\alpha,\beta,\omega)}{\mu_{\epsilon\eta}(\alpha,\beta,\omega)}}x^2\Big] H_n\Big(\sqrt[4]{\frac{\nu_{\epsilon\eta}(\alpha,\beta,\omega)}{\mu_{\epsilon\eta}(\alpha,\beta,\omega)}}x\Big). \tag{5.5}$$

where the $H_n\Big(\sqrt[4]{\frac{\mu_{\epsilon\eta}(\alpha,\beta,\omega)}{\nu_{\epsilon\eta}(\alpha,\beta,\omega)}}x\Big)$ are Hermite polynomials.

In the momentum representation the Hamiltonian (5.1) leads to the differential equation

$$\frac{1}{2}\Big[\nu_{\epsilon\eta}(\alpha,\beta,\omega)\frac{d^2}{dp^2} + E - \mu_{\epsilon\eta}(\alpha,\beta,\omega)p^2\Big]\varphi(p) = 0. \tag{5.6}$$

which admits for $E_n = \left(m + \frac{1}{2}\right)\sqrt{\omega^2 - 4\alpha\beta}$, the solution

$$\psi_m(p) = \frac{(-i)^m}{\sqrt{2^m m!}} \exp\Big[-\frac{1}{2}\sqrt{\frac{\mu_{\epsilon\eta}(\alpha,\beta,\omega)}{\nu_{\epsilon\eta}(\alpha,\beta,\omega)}}p^2\Big] H_m\Big(\sqrt[4]{\frac{\mu_{\epsilon\eta}(\alpha,\beta,\omega)}{\nu_{\epsilon\eta}(\alpha,\beta,\omega)}}p\Big). \tag{5.7}$$

At to point when we consider the eigenstates of the Hamiltonian (5.1) in both representations (cfr (5.5) and (5.7)), we found that they are connected to the eigenstates of the non-Hamiltonian quadratic Hamiltonian by the similarity transformation $\mathbf{S}$. In addition, the structure of the eigenstates require both $\mu_{\epsilon\eta}(\alpha,\beta,\omega)$ and $\nu_{\epsilon\eta}(\alpha,\beta,\omega)$ to be positive. In the next section a more specific analysis will be conducted in that direction. Such analysis may lead to preferential selections of the parameter η .

5.2 How to select a proper setting of the parameter η .

In the previous chapter we have come to the conclusion that there exist as many proper ways of obtaining unique setting of the parameter η in the metric as the number of irreducible sets we can find for the system represented by the Hamiltonian (2.2). We have also established that the parameter belongs to the interval $-\frac{\epsilon}{2} \leq \eta \leq \frac{\epsilon}{2}$. This interval should be considered as a segment on which each observable can be classified with respect to its degrees of dependence on the position operator $\hat{\mathbf{x}}$ and the momentum operator $\hat{\mathbf{p}}$. For example any operator exclusively depending on the momentum will be an observable only for $\eta = -\frac{\epsilon}{2}$, and any operator exclusively depending on the position will be an observable only for $\eta = \frac{\epsilon}{2}$. Similarly to the standard quantum theory there exist operators depending on $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ which are not observables. For those operators, η does not exist. As example we consider the boson operator $\mathbf{a} = \sqrt{\frac{\omega}{2}}\hat{\mathbf{x}} + \frac{i}{\sqrt{2\omega}}\hat{\mathbf{p}}$, we require

$$\mathbf{aT} = \mathbf{Ta}^{\dagger}. \tag{5.8}$$

We first multiply from the left both sides of (5.8) by $\mathbf{S}^{-1}$, and then from the right. We obtain

$$\begin{aligned}
\mathbf{S}^{-1}\mathbf{aS} &= \mathbf{Sa}^{\dagger}\mathbf{S}^{-1} && (5.9)\\
\left[\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right]\mathbf{a} + \frac{2\eta}{\theta}\sinh\theta\mathbf{a}^{\dagger} &= \frac{2\eta}{\theta}\sinh\theta\mathbf{a} + \left[\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right]\mathbf{a}^{\dagger} && (5.10)\\
\left[\cosh\theta + \frac{\epsilon - 2\eta}{\theta}\sinh\theta\right]\mathbf{a} &= \left[\cosh\theta + \frac{\epsilon - 2\eta}{\theta}\sinh\theta\right]\mathbf{a}^{\dagger}. && (5.11)
\end{aligned}$$

We obtain an ambiguous equation

$$\cosh\theta + \frac{\epsilon - 2\eta}{\theta}\sinh\theta = 0 \tag{5.12}$$

from which there are no real solutions in parameters η and ϵ. This example illustrates the use of metric is consistent with the definition of the boson operator $\mathbf{a}$ as we know from the standard quantum theory.

More examples considering position operator and momentum operator will be considered in the last section of this chapter. For now we want to explore possible leads on the choice of the parameter η.

5.2.1 Implications of $\eta = 0$ on the physical aspects

The parameter $\eta = 0$ corresponds to the commutation of the metric $\mathbf{T}$ with all the elements of the irreducible set $\{\mathbf{H}, \hat{\mathbf{n}}\}$. The metric $\mathbf{T}$ reduces to the form $\mathbf{T} = \left(\frac{\alpha}{\beta}\right)^{\frac{\hat{\mathbf{n}}}{2}}$, and it induces the similarity transformation $\mathbf{S}$ that maps the non-Hermitian quadratic Hamiltonian $\mathbf{H}$ (2.2) on the Hermitian quadratic Hamiltonian $\tilde{\mathbf{H}}$

$$\tilde{\mathbf{H}} = \omega\left(\mathbf{a}^{\dagger}\mathbf{a} + \frac{1}{2}\right) + \sqrt{\alpha\beta}\left(\mathbf{a}^2 + \mathbf{a}^{\dagger 2}\right). \tag{5.13}$$

The canonical representation (5.1) becomes

$$\tilde{\mathbf{H}} = \frac{1}{2\omega}[\omega - 2\sqrt{\alpha\beta}]\hat{\mathbf{p}}^2 + \frac{\omega}{2}[\omega + 2\sqrt{\alpha\beta}]\hat{\mathbf{x}}^2, \tag{5.14}$$

and in position representation the eigenstates are

$$\psi_n(x) = \frac{(-i)^n}{\sqrt{2^n n!}} \exp\Big[-\frac{1}{2}\sqrt{\frac{\omega + 2\sqrt{\alpha\beta}}{\omega - 2\sqrt{\alpha\beta}}}x^2\Big] H_n\Big(\sqrt[4]{\frac{\omega + 2\sqrt{\alpha\beta}}{\omega - 2\sqrt{\alpha\beta}}}x\Big), \tag{5.15}$$

By observing the eigenstates (5.15), we can see that the requirement $\omega > 2\sqrt{\alpha\beta}$ is just another form of requiring the reality of the eigen energies.

5.2.2 Implications of $\eta = \frac{\epsilon}{2}$ on the physical aspects

The parameter $\eta = \frac{\epsilon}{2}$ corresponds to the commutation of the metric $\mathbf{T}$ with all the elements of the irreducible set $\{\mathbf{H}, \hat{\mathbf{x}}\}$. The metric $\mathbf{T}$ reduces to the form $\mathbf{T} = \exp\Big(-\frac{1}{2}\frac{\alpha-\beta}{\omega-\alpha-\beta}\hat{\mathbf{x}}^2\Big)$, and it induces the similarity transformation $\mathbf{S}$ that maps the on-Hermitian quadratic Hamiltonian $\mathbf{H}$ (2.2) on the Hermitian quadratic Hamiltonian $\tilde{\mathbf{H}}$

$$\tilde{\mathbf{H}} = \Big[\omega + \frac{(\alpha-\beta)^2}{2(\omega-\alpha-\beta)}\Big]\Big(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\Big) + \frac{1}{4}\Big[\frac{\omega^2 - 4\alpha\beta}{\omega-\alpha-\beta} - (\omega-\alpha-\beta)\Big]\Big(\mathbf{a}^2 + \mathbf{a}^{\dagger 2}\Big). \tag{5.16}$$

The canonical representation (5.1) becomes

$$\tilde{\mathbf{H}} = \frac{1}{2}\Big(\frac{\omega-\alpha-\beta}{\omega}\hat{\mathbf{p}}^2 + \frac{\omega^2-4\alpha\beta}{\omega-\alpha-\beta}\,\omega\hat{\mathbf{x}}^2\Big), \tag{5.17}$$

and in position representation the eigenstates are

$$\psi_n(x) = \frac{(-i)^n}{\sqrt{2^n n!}} \exp\Big[-\frac{1}{2}\frac{\omega\sqrt{\omega^2-4\alpha\beta}}{\omega-\alpha-\beta}x^2\Big] H_n\Big(\sqrt{\frac{\omega\sqrt{\omega^2-4\alpha\beta}}{\omega-\alpha-\beta}}x\Big), \tag{5.18}$$

5.2.3 Implications of $\eta = -\frac{\epsilon}{2}$ on the physical aspects

The parameter $\eta = -\frac{\epsilon}{2}$ corresponds to the commutation of the metric $\mathbf{T}$ with all the elements of the irreducible set $\{\mathbf{H}, \hat{\mathbf{p}}\}$. The metric $\mathbf{T}$ reduces to the form $\mathbf{T} = \exp\Big(\frac{1}{2}\frac{\alpha-\beta}{\omega+\alpha+\beta}\hat{\mathbf{p}}^2\Big)$, and it induces the similarity transformation $\mathbf{S}$ that maps the non-Hermitian quadratic Hamiltonian $\mathbf{H}$ (2.2) on the Hermitian quadratic Hamiltonian $\tilde{\mathbf{H}}$

$$\tilde{\mathbf{H}} = \Big[\omega + \frac{(\alpha-\beta)^2}{2(\omega+\alpha+\beta)}\Big]\Big(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\Big) + \frac{1}{4}\Big[\frac{\omega^2 - 4\alpha\beta}{\omega+\alpha+\beta} - (\omega+\alpha+\beta)\Big]\Big(\mathbf{a}^2 + \mathbf{a}^{\dagger 2}\Big). \tag{5.19}$$

The canonical representation (5.1) becomes

$$\tilde{\mathbf{H}} = \frac{1}{2}\Big(\frac{\omega^2-4\alpha\beta}{\omega(\omega+\alpha+\beta)}\hat{\mathbf{p}}^2 + (\omega+\alpha+\beta)\,\omega\hat{\mathbf{x}}^2\Big), \tag{5.20}$$

and in momentum representation the eigenstates are

$$\varphi_m(p) = \frac{(-i)^m}{\sqrt{2^m m!}} \exp\Big[-\frac{1}{2}\frac{\sqrt{\omega^2-4\alpha\beta}}{\omega(\omega+\alpha+\beta)}p^2\Big] H_m\Big(\sqrt{\frac{\sqrt{\omega^2-4\alpha\beta}}{\omega(\omega+\alpha+\beta)}}\,p\Big), \tag{5.21}$$

The canonical form of the Hermitian quadratic Hamiltonian $\tilde{\mathbf{H}}$ and the structure of eigenstates, suggest that they may be further analysis with respect to the sign of the coefficients $\mu_{\epsilon\eta}(\alpha,\beta,\omega)$ for the position representation and $\nu_{\epsilon\eta}(\alpha,\beta,\omega)$ for the momentum representation. In fact $\mu_{\epsilon\eta}(\alpha,\beta,\omega)$ in (5.6) takes the form $\mu_{\epsilon\eta}(\alpha,\beta,\omega) = \omega - 2\sqrt{\alpha\beta}$ in (5.15) and (5.16) while it takes the form $\mu_{\epsilon\eta}(\alpha,\beta,\omega) = \omega-\alpha-\beta$ in (5.18) and (5.19) suggest the condition $\mu_{\epsilon\eta}(\alpha,\beta,\omega) > 0$ may contain an additional information for a preferential choice of the parameter η with regard to the size of α and β in addition to the requirement $\omega^2-4\alpha\beta$.

An observation on these requirement for $\mu_{\epsilon\eta}(\alpha,\beta,\omega)$ in both case $\eta = 0$ and $\eta = \frac{\epsilon}{2}$ shows that if it is obvious that the condition $\omega > 2\sqrt{\alpha\beta}$ is a repetition, it is also true that there condition $\omega > \alpha+\beta$ is weaker than $\omega^2-4\alpha\beta$. Therefore, we can conclude that the physical system presented by the non-Hermitian quadratic Hamiltonian (2.2)can be equivalently studied from each one of the point of view as far as the parameter η lies in the interval $-\frac{\epsilon}{2} \le \eta \le \frac{\epsilon}{2}$.

Another interesting fact is that the parametric dependence of the quadratic Hermitian Hamiltonian $\tilde{\mathbf{H}}$ introduces the connection between several sets of eigenstates whose the basis transformation induces only the change of the argument. Such a behaviour can also be extended to the Hermite polynomials within the eigenstates. This feature presents an interesting picture where the quantum mechanical system can travel between several numbers of physical states with the change of the parameter η while keeping its eigen-energies the same.

The choice of a preferential setting for the value of η must only be related to the nature of observables one need to use for the study of the physical system represented by (2.2).

5.3 Examples

5.3.1 Example 1: Particular case $\alpha = 0$ (or $\beta = 0$)

We have seen that the correlation between metric and observables must guide the choice of a set of some particular values for the free parameter η . The following example shows how the metric $\mathbf{T}$ behave with respect to the non-Hermitian quadratic Hamiltonian (2.2) when α (or β) is zero.

As long as the parameter η fulfils

$$\frac{1}{\theta}\tanh 2\theta = -\frac{\beta}{\beta\epsilon - 2\omega\eta},$$

the metric $\mathbf{T}$ hermitizes the Hamiltonian (2.2) with respect to the T-inner product, with the only problem being the case $\eta = 0$ For these values of η in the interval $-\frac{\epsilon}{2} \leq \eta \leq \frac{\epsilon}{2}$. In other words, for

$$\mathbf{H} = \omega\left(\mathbf{a}^{\dagger}\mathbf{a} + \frac{1}{2}\right) + \beta\mathbf{a}^{\dagger 2}, \tag{5.22}$$

the metric (2.40) for $\eta = 0$:

$$\mathbf{T} = \mathbf{S}^{\dagger}\mathbf{S} = e^{2\epsilon\mathbf{a}^{\dagger}\mathbf{a}}, \tag{5.23}$$

cannot hermitizes the non-Hermitian quadratic Hamiltonian. But for all other values of the parameter η, the eigenvalues, the right hand eigenstates, and the left hand eigenstates simplify into

$$\begin{aligned} E_n &= \left(n + \frac{1}{2}\right)\omega, & (5.24)\\ |n\rangle_R &= \mathbf{S}^{-1}|n\rangle, & (5.25)\\ |m\rangle_L &= \mathbf{S}^{\dagger}|n\rangle. & (5.26)\end{aligned}$$

Since the Bogoliubov transformation (3.8) has become the unity operator $\mathbf{I}$, the eigenstates $|n_b\rangle$ are just $|n\rangle$.

5.3.2 Example 2: Observables

We consider the diagonal Hamiltonian $\bar{\mathbf{H}}$ (3.9). since

$$\bar{\mathbf{H}} = \frac{1}{2}\left(\bar{\mathbf{p}}^2 + \Omega^2\bar{\mathbf{x}}^2\right), \tag{5.27}$$

we can deduce the position operator $\bar{\mathbf{x}}$ and the momentum operator $\bar{\mathbf{p}}$ and obtain

$$\begin{aligned} \bar{\mathbf{x}} &= \frac{1}{\sqrt{2\Omega}}\left(\mathbf{b} + \mathbf{b}^{\dagger}\right) & (5.28)\\ \bar{\mathbf{p}} &= i\sqrt{\frac{\Omega}{2}}\left(\mathbf{b}^{\dagger} - \mathbf{b}\right), & (5.29)\end{aligned}$$

where the boson creation and annihilation operators are defined in (3.3). Reversing the Bogoliubov transformation (3.4)

$$\begin{aligned} \tilde{\mathbf{H}} &= \mathbf{B}^{\dagger}\bar{\mathbf{H}}\mathbf{B}\\ &= \frac{1}{2}\left(\mathbf{B}^{\dagger}\bar{\mathbf{p}}^2\mathbf{B} + \Omega^2\mathbf{B}^{\dagger}\bar{\mathbf{x}}^2\mathbf{B}\right)\\ &= \frac{1}{2}\left(\tilde{\mathbf{p}}^2 + \Omega^2\tilde{\mathbf{x}}^2\right), & (5.30)\end{aligned}$$

we obtain the position observables $\tilde{\mathbf{x}}$ and the momentum observables $\tilde{\mathbf{p}}$. These two observables are the result of the hermitization of some non-Hermitian position operator $\mathbf{x}$ and non-Hermitian momentum operator $\mathbf{p}$; in other words

$$\mathbf{x} = \mathbf{S}^{-1}\tilde{\mathbf{x}}\mathbf{S} = \frac{1}{\sqrt{2\Omega}}\left(\mathbf{S}^{-1}\mathbf{a}\mathbf{S} + \mathbf{S}^{-1}\mathbf{a}^{\dagger}\mathbf{S}\right) \tag{5.31}$$

$$\mathbf{p} = \mathbf{S}^{-1}\tilde{\mathbf{p}}\mathbf{S} = i\sqrt{\frac{\Omega}{2}}\left(\mathbf{S}^{-1}\mathbf{a}^{\dagger}\mathbf{S} - \mathbf{S}^{-1}\mathbf{a}\mathbf{S}\right). \tag{5.32}$$

After application of the infinitesimal form of the similarity transformation $\mathbf{S}$ we obtain respectively

$$\mathbf{x} = \sqrt{\frac{\omega}{\Omega}}\cosh\theta\hat{\mathbf{x}} + \frac{i}{\sqrt{\omega\Omega}}\frac{\epsilon - 2\eta}{\theta}\sinh\theta\hat{\mathbf{p}}, \tag{5.33}$$

$$\mathbf{p} = \sqrt{\frac{\Omega}{\omega}}\cosh\theta\hat{\mathbf{p}} - i\sqrt{\omega\Omega}\frac{\epsilon + 2\eta}{\theta}\sinh\theta\hat{\mathbf{x}}. \tag{5.34}$$

We need to evaluate standard deviation of each one of these $\mathcal{PT}$-symmetric quantum theory observables. In fact we call them $\mathcal{PT}$-symmetric quantum theory observables because they are all Hermitian with respect to the T-inner product even though at different values of the parameter η; in other words

$$\mathbf{x}\mathbf{T} = \mathbf{T}\mathbf{x}^{\dagger}, \tag{5.35}$$

for $\eta = \frac{\epsilon}{2}$, and

$$\mathbf{p}\mathbf{T} = \mathbf{T}\mathbf{p}^{\dagger}, \tag{5.36}$$

for $-\eta = \frac{\epsilon}{2}$. More details proving the values of η for the requirements (5.35) and (5.36) are presented in Appendix E.

We want to evaluate the standard deviation in the eigenstate $|n\rangle_R$ using the metric's $\mathbf{T}$ parameter $\eta = \frac{\epsilon}{2}$. In that case, these two $\mathcal{PT}$-symmetric quantum theory observables become

$$\mathbf{x} = \sqrt{\frac{\omega}{\Omega}}\hat{\mathbf{x}}, \tag{5.37}$$

$$\mathbf{p} = \sqrt{\frac{\Omega}{\omega}}\hat{\mathbf{p}} + i\sqrt{\omega\Omega}\frac{\alpha - \beta}{\omega - \alpha - \beta}\hat{\mathbf{x}}. \tag{5.38}$$

By definition the standard deviation is given by

$$\Delta x = \sqrt{\langle\mathbf{x}^2\rangle - \langle\mathbf{x}\rangle^2}. \tag{5.39}$$

From the definition (2.3), it appears that Δx will only receive the contribution from $\mathbf{x}^2$, which contribution is

$$\begin{aligned} \langle \mathbf{x}^2 \rangle &= {}_R\langle n|\mathbf{T}\mathbf{x}^2|n\rangle_R \\ &= \frac{\omega}{\Omega}{}_L\langle n|\hat{\mathbf{x}}^2|n\rangle_R \\ &= \left(n+\frac{1}{2}\right)\frac{1}{\Omega} \end{aligned} \tag{5.40}$$

Therefore the standard deviation is

$$\Delta x = \left(n+\frac{1}{2}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\Omega}} \tag{5.41}$$

In practice, the standard quantum theory uses the evaluation of Δx combined with Δp for the evaluation of the Heisenberg uncertainty product. In this example it will be interesting to see what happen for the Heisenberg uncertainty product in $\mathcal{PT}$-symmetric quantum theory? For this purpose we need to evaluate the momentum standard deviation Δp . As for Δx , the contribution in this case comes from $\mathbf{p}^2$, which is

$$\begin{aligned} \mathbf{p}^2 &= \mathbf{p}^\dagger\mathbf{p} \\ &= \frac{\Omega}{\omega}\hat{\mathbf{p}}^2 + \frac{(\alpha-\beta)^2}{(\omega-\alpha-\beta)^2}\omega\Omega\hat{\mathbf{x}}^2 - \sqrt{\omega\Omega}\frac{\alpha-\beta}{\omega-\alpha-\beta}. \end{aligned} \tag{5.42}$$

Therefore

$$\begin{aligned} \langle \mathbf{p}^2 \rangle &= {}_R\langle n|\mathbf{T}\mathbf{p}^2|n\rangle_R \\ &= \frac{\Omega}{\omega}{}_R\langle n|\mathbf{T}\hat{\mathbf{p}}^2|n\rangle_R + \frac{(\alpha-\beta)^2}{(\omega-\alpha-\beta)^2}\omega\Omega_R\langle n|\mathbf{T}\hat{\mathbf{x}}^2|n\rangle_R - \sqrt{\omega\Omega}\frac{\alpha-\beta}{\omega-\alpha-\beta}{}_R\langle n|\mathbf{T}|n\rangle_R \\ &= \frac{\Omega}{\omega}{}_R\langle n|\mathbf{T}\hat{\mathbf{p}}^2|n\rangle_R + \left(n+\frac{1}{2}\right)\frac{(\alpha-\beta)^2}{(\omega-\alpha-\beta)^2}\Omega - \sqrt{\omega\Omega}\frac{\alpha-\beta}{\omega-\alpha-\beta}, \end{aligned} \tag{5.43}$$

On the other hand,

$$\begin{aligned} \frac{\Omega}{\omega}{}_R\langle n|\mathbf{T}\hat{\mathbf{p}}^2|n\rangle_R &= \frac{\Omega}{\omega}{}_L\langle n|\hat{\mathbf{p}}^2|n\rangle_R \\ &= \frac{\Omega}{\omega}\langle n_b|\mathbf{B}\mathbf{S}\hat{\mathbf{p}}^2\mathbf{S}^{-1}\mathbf{B}^\dagger|n_b\rangle \end{aligned} \tag{5.44}$$

and for $\eta = \frac{\epsilon}{2}$ (For more details see Appendix E.)

$$\mathbf{S}\hat{\mathbf{p}}\mathbf{S}^{-1} = \hat{\mathbf{p}} - i\omega\frac{\alpha-\beta}{\omega-\alpha-\beta}\hat{\mathbf{x}}, \tag{5.45}$$

therefore

$$\mathbf{S}\hat{\mathbf{p}}^2\mathbf{S}^{-1} = \hat{\mathbf{p}}^2 - \omega^2\frac{(\alpha-\beta)^2}{(\omega-\alpha-\beta)^2}\hat{\mathbf{x}}^2 - i\omega\frac{\alpha-\beta}{\omega-\alpha-\beta}\Big(\hat{\mathbf{x}}\hat{\mathbf{p}}+\hat{\mathbf{p}}\hat{\mathbf{x}}\Big). \tag{5.46}$$

The matrix element $\frac{\Omega}{\omega}{}_R\langle n|\mathbf{T}\hat{\mathbf{p}}^2|n\rangle_R$ becomes

$$\begin{aligned}\frac{\Omega}{\omega}{}_R\langle n|\mathbf{T}\hat{\mathbf{p}}^2|n\rangle_R &= \frac{\Omega}{\omega}\langle n_b|\mathbf{B}\hat{\mathbf{p}}^2\mathbf{B}^\dagger|n_b\rangle - \frac{(\alpha-\beta)^2}{(\omega-\alpha-\beta)^2}\omega\Omega\langle n_b|\mathbf{B}\hat{\mathbf{x}}^2\mathbf{B}^\dagger|n_b\rangle \\ &\quad -i\frac{\alpha-\beta}{\omega-\alpha-\beta}\sqrt{\omega\Omega}\langle n_b|\mathbf{B}\Big(\hat{\mathbf{x}}\hat{\mathbf{p}}+\hat{\mathbf{p}}\hat{\mathbf{x}}\Big)\mathbf{B}^\dagger|n_b\rangle \\ &= \Big(n+\frac{1}{2}\Big)\Omega - \Big(n+\frac{1}{2}\Big)\frac{(\alpha-\beta)^2}{(\omega-\alpha-\beta)^2}\omega\Omega + \sqrt{\omega\Omega}\frac{\alpha-\beta}{\omega-\alpha-\beta}\end{aligned} \tag{5.47}$$

Combining (5.43) and (5.47), we obtain

$$\Delta p = \sqrt{\Big(n+\frac{1}{2}\Big)\Omega}. \tag{5.48}$$

The Heisenberg uncertainty product is therefore given by

$$\Delta x\Delta p = \Big(n+\frac{1}{2}\Big) \tag{5.49}$$

In the ground state $n = 0$ this product becomes

$$\Delta x\Delta p = \frac{1}{2} \tag{5.50}$$

We recall that $\hbar = 1$. The expression (5.49) is the Heisenberg uncertainty for the physical system represented by the non-Hermitian quadratic Hamiltonian (2.2) in $\mathcal{PT}$-symmetric quantum theory. This product appears as an expansion is the same with the one of the standard quantum theory.

The same evaluation can be done with $\eta = -\frac{\epsilon}{2}$ or any other value of η for $-\frac{\epsilon}{2} \leq \eta \leq \frac{\epsilon}{2}$.

CHAPTER 6
CONCLUSION

This thesis has presented the metrics formulation of the non-Hermitian PT-symmetric quantum theory as both a complete extension and proper alternative to the standard quantum theory. As a complete extension , we have seen how the concept of hermiticity can be extended when one can find a metric operator which hemitizes a non-Hermitian Hamiltonian. The use of such a metric is facilitated by the space-time reflection symmetry of the non-Hermitian Hamiltonian and both the right hand and the left hand eigenstates. As a proper alternative to the standard quantum theory, we consider the case where dealing with standard quantum theory becomes difficult or complicated, the PT-symmetric quantum theory. These work [5] by C.M. Bender et al, and [33] by D. Janssen, and P. Schuck present good examples for such case. In fact, C Bender et al show that the $\mathcal{O}(g^4)$ contribution on the ground state for the the $i\hat{\mathbf{x}}^3$ is much easier to evaluate using non-hermiticity rather than hermiticity and D. Janssen, and P. Schuck show that the translational invariance in the Hartree-Fock theory for finite Fermi system can be only restored using the non-hermiticity.

The study of the non-Hermitian quadratic Hamiltonian using the metric as constructed in this thesis shows that there remains the question on what choice can we make to obtain a unique physical answer? In fact since the Hermitian correspondent $\tilde{\mathbf{H}}$ depends on the free parameter, the non Hermitian system represented by (2.2) corresponds to a family of Hermitian Hamiltonian $\tilde{\mathbf{H}}$ each one involving a set of eigenstates deffering from another by the argument. Furthermore, the existence of this family of $\tilde{\mathbf{H}}$ has a direct consequence on the probability density and sometime may affect the expectation values of observables as presented in this thesis for the amplitude of transition.

The consistent formulation of mathematical framework can be derived from the use of metric in non-Hermitian quantum theory. The choice fixing the metric dependence on the free parameter affects some physical aspects.

APPENDIX A

We need to evaluate $e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}}$ and $e^{\mathbf{A}}\mathbf{a}^{\dagger}e^{-\mathbf{A}}$. Using the Baker-Campbell-Haussdorf theorem, we can write them in infinitesimal form as,

$$
\begin{aligned}
e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}} &= \sum_{n=0}^{\infty}\frac{1}{n!}(\mathbf{A},\,\mathbf{a})_{\mathbf{n}} \qquad \text{(A.1)}\\
e^{\mathbf{A}}\mathbf{a}^{\dagger}e^{-\mathbf{A}} &= \sum_{n=0}^{\infty}\frac{1}{n!}(\mathbf{A},\,\mathbf{a}^{\dagger})_{n} \qquad \text{(A.2)}
\end{aligned}
$$

With

$$
\begin{cases}
(\mathbf{A},\,\mathbf{a})_n = [\mathbf{A},\,(\mathbf{A},\,\mathbf{a})_{n-1}] \\
\qquad\qquad\qquad\qquad ; \\
(\mathbf{A},\,\mathbf{a})_0 = \mathbf{a}
\end{cases} \qquad \text{(A.3)}
$$

and

$$
\begin{cases}
(\mathbf{A},\,\mathbf{a}^{\dagger})_n = [\mathbf{A},\,(\mathbf{A},\,\mathbf{a}^{\dagger})_{n-1}] \\
\qquad\qquad\qquad\qquad ; \\
(\mathbf{A},\,\mathbf{a}^{\dagger})_0 = \mathbf{a}^{\dagger}
\end{cases} \qquad \text{(A.4)}
$$

On the other side,

$$
\begin{aligned}
[\mathbf{A},\mathbf{a}] &= [\eta\,\mathbf{a}^2 + \eta^*\,\mathbf{a}^{\dagger 2} + \epsilon\,\mathbf{a}^{\dagger}\mathbf{a},\,\mathbf{a}]\\
&= \eta[\mathbf{a}^2,\mathbf{a}] + \eta^*[\mathbf{a}^{\dagger 2},\mathbf{a}] + \epsilon[\mathbf{a}^{\dagger}\mathbf{a},\mathbf{a}]\\
&= 2\eta\mathbf{a}[\mathbf{a},\mathbf{a}] + 2\eta^*\mathbf{a}^{\dagger}[\mathbf{a}^{\dagger},\mathbf{a}] + \epsilon\left([\mathbf{a}^{\dagger},\mathbf{a}]\mathbf{a} + \mathbf{a}^{\dagger}[\mathbf{a},\mathbf{a}]\right)\\
&= -\epsilon\,\mathbf{a} - 2\eta^*\mathbf{a}^{\dagger} \qquad \text{(A.5)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathbf{A},\mathbf{a}^{\dagger}] &= [\eta\,\mathbf{a}^2 + \eta^*\,\mathbf{a}^{\dagger 2} + \epsilon\,\mathbf{a}^{\dagger}\mathbf{a},\,\mathbf{a}^{\dagger}]\\
&= \eta[\mathbf{a}^2,\mathbf{a}^{\dagger}] + \eta^*[\mathbf{a}^{\dagger 2},\mathbf{a}^{\dagger}] + \epsilon[\mathbf{a}^{\dagger}\mathbf{a},\mathbf{a}^{\dagger}]\\
&= 2\eta\mathbf{a}[\mathbf{a},\mathbf{a}^{\dagger}] + 2\eta^*\mathbf{a}^{\dagger}[\mathbf{a}^{\dagger},\mathbf{a}^{\dagger}] + \epsilon\left([\mathbf{a}^{\dagger},\mathbf{a}^{\dagger}]\mathbf{a} + \mathbf{a}^{\dagger}[\mathbf{a},\mathbf{a}^{\dagger}]\right)\\
&= 2\eta\mathbf{a} + \epsilon\,\mathbf{a}^{\dagger} \qquad \text{(A.6)}
\end{aligned}
$$

Let's expand $(\mathbf{A},\,\mathbf{a})_n$

$$
\begin{aligned}
(\mathbf{A},\,\mathbf{a})_0 &= \mathbf{a}\\
(\mathbf{A},\,\mathbf{a})_1 &= [\mathbf{A},\,(\mathbf{A},\,\mathbf{a})_0] = -\epsilon\,\mathbf{a} - 2\eta^*\mathbf{a}^\dagger\\
(\mathbf{A},\,\mathbf{a})_2 &= [\mathbf{A},\,(\mathbf{A},\,\mathbf{a})_1] = -\epsilon\,[\mathbf{A},\mathbf{a}] - 2\eta^*[\mathbf{A},\mathbf{a}^\dagger] = \left(\epsilon^2 - 4|\eta|^2\right)\mathbf{a}\\
(\mathbf{A},\,\mathbf{a})_3 &= [\mathbf{A},\,(\mathbf{A},\,\mathbf{a})_2] = -\left(\epsilon^2 - 4|\eta|^2\right)\left(\epsilon\,\mathbf{a} + 2\eta^*\mathbf{a}^\dagger\right)\\
(\mathbf{A},\,\mathbf{a})_4 &= [\mathbf{A},\,(\mathbf{A},\,\mathbf{a})_3] = \left(\epsilon^2 - 4|\eta|^2\right)^2\mathbf{a}\\
(\mathbf{A},\,\mathbf{a})_5 &= [\mathbf{A},\,(\mathbf{A},\,\mathbf{a})_4] = -\left(\epsilon^2 - 4|\eta|^2\right)^2\mathbf{a}\left(\epsilon\,\mathbf{a} + 2\eta^*\mathbf{a}^\dagger\right)\\
(\mathbf{A},\,\mathbf{a})_6 &= [\mathbf{A},\,(\mathbf{A},\,\mathbf{a})_5] = \left(\epsilon^2 - 4|\eta|^2\right)^3\mathbf{a}
\end{aligned}
$$

It follows that in general

$$
(\mathbf{A},\,\mathbf{a})_n = \left\langle
\begin{array}{ll}
\left(\epsilon^2 - 4|\eta|^2\right)^{\frac{n}{2}}\mathbf{a} & \textit{for n even}\\
 & \\
-\left(\epsilon^2 - 4|\eta|^2\right)^{\frac{n-1}{2}}\left(\epsilon\,\mathbf{a} + \mathbf{2}\eta^*\mathbf{a}^\dagger\right) & \textit{for n odd}
\end{array}\right. ; \tag{A.7}
$$

It follows that

$$
\begin{aligned}
e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}} &= \sum_{n=0,2,4,\ldots}^{\infty}\frac{1}{2(\frac{n}{2})!}\left(\epsilon^2 - 4|\eta|^2\right)^{\frac{n}{2}}\mathbf{a}\\
&\quad - \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{(2n-1)!!}\left(\epsilon^2 - 4|\eta|^2\right)^{\frac{n-1}{2}}\left(\epsilon\,\mathbf{a} + 2\eta^*\mathbf{a}^\dagger\right)\\
&= \sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\epsilon^2 - 4|\eta|^2\right)^{k}\mathbf{a}\\
&\quad - \sum_{l=0}^{\infty}\frac{1}{(2l+1)!}\left(\epsilon^2 - 4|\eta|^2\right)^{l}\left(\epsilon\,\mathbf{a} + 2\eta^*\mathbf{a}^\dagger\right)
\end{aligned} \tag{A.8}
$$

with $n = 2k$ in the first series and $n = 2l + 1$ in the second. Using the Taylor series for hyperbolic functions we can write;

$$
\begin{aligned}
e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}} &= \left[\cosh\sqrt{\epsilon^2 - 4|\eta|^2} - \frac{\epsilon}{\sqrt{\epsilon^2 - 4|\eta|^2}}\sinh\sqrt{\epsilon^2 - 4|\eta|^2}\right]\mathbf{a}\\
&\quad - \frac{2\eta^*}{\sqrt{\epsilon^2 - 4|\eta|^2}}\sinh\sqrt{\epsilon^2 - 4|\eta|^2}\,\mathbf{a}^\dagger
\end{aligned} \tag{A.9}
$$

We also need to expand $(\mathbf{A},\, \mathbf{a}^\dagger)_n$

$$
\begin{aligned}
(\mathbf{A},\, \mathbf{a}^\dagger)_0 &= \mathbf{a}^\dagger \\
(\mathbf{A},\, \mathbf{a}^\dagger)_1 &= [\mathbf{A},\, (\mathbf{A},\, \mathbf{a}^\dagger)_0] = 2\eta\mathbf{a} + \epsilon\,\mathbf{a}^\dagger \\
(\mathbf{A},\, \mathbf{a}^\dagger)_2 &= [\mathbf{A},\, (\mathbf{A},\, \mathbf{a}^\dagger)_1] = 2\eta[\mathbf{A},\mathbf{a}] + \epsilon\,[\mathbf{A},\mathbf{a}^\dagger] = \left(\epsilon^2 - 4|\eta|^2\right)\mathbf{a}^\dagger \\
(\mathbf{A},\, \mathbf{a}^\dagger)_3 &= [\mathbf{A},\, (\mathbf{A},\, \mathbf{a})_2] = -\left(\epsilon^2 - 4|\eta|^2\right)\left(2\eta\mathbf{a} + \epsilon\,\mathbf{a}^\dagger\right) \\
(\mathbf{A},\, \mathbf{a}^\dagger)_4 &= [\mathbf{A},\, (\mathbf{A},\, \mathbf{a})_3] = \left(\epsilon^2 - 4|\eta|^2\right)^2\mathbf{a}^\dagger \\
(\mathbf{A},\, \mathbf{a}^\dagger)_5 &= [\mathbf{A},\, (\mathbf{A},\, \mathbf{a})_4] = -\left(\epsilon^2 - 4|\eta|^2\right)^2\left(2\eta\mathbf{a} + \epsilon\,\mathbf{a}^\dagger\right) \\
(\mathbf{A},\, \mathbf{a}^\dagger)_6 &= [\mathbf{A},\, (\mathbf{A},\, \mathbf{a})_5] = \left(\epsilon^2 - 4|\eta|^2\right)^3\mathbf{a}^\dagger
\end{aligned}
$$

It follows that in general

$$
(\mathbf{A},\, \mathbf{a}^\dagger)_n = \left\langle
\begin{array}{ll}
\left(\epsilon^2 - 4|\eta|^2\right)^{\frac{n}{2}}\mathbf{a}^\dagger & for\ n\ even \\
\\
\left(\epsilon^2 - 4|\eta|^2\right)^{\frac{n-1}{2}}\left(2\eta\mathbf{a} + \epsilon\,\mathbf{a}^\dagger\right) & for\ n\ odd
\end{array}
\right. ; \tag{A.10}
$$

It follows that

$$
\begin{aligned}
e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}} &= \sum_{n=0,2,4,\ldots}^{\infty}\frac{1}{2(\frac{n}{2})!}\left(\epsilon^2 - 4|\eta|^2\right)^{\frac{n}{2}}\mathbf{a} \\
&\quad + \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{(2n-1)!!}\left(\epsilon^2 - 4|\eta|^2\right)^{\frac{n-1}{2}}\left(2\eta\mathbf{a} + \epsilon\,\mathbf{a}^\dagger\right) \\
&= \sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\epsilon^2 - 4|\eta|^2\right)^{k}\mathbf{a} \\
&\quad + \sum_{l=0}^{\infty}\frac{1}{(2l+1)!}\left(\epsilon^2 - 4|\eta|^2\right)^{l}\left(2\eta\mathbf{a} + \epsilon\,\mathbf{a}^\dagger\right)
\end{aligned} \tag{A.11}
$$

with $n = 2k$ in the first series and $n = 2l+1$ in the second. Using the Taylor series for hyperbolic functions we can write;

$$
\begin{aligned}
e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}} &= \frac{2\eta}{\sqrt{\epsilon^2 - 4|\eta|^2}}\sinh\sqrt{\epsilon^2 - 4|\eta|^2}\,\mathbf{a} \\
&\quad + \left[\cosh\sqrt{\epsilon^2 - 4|\eta|^2} + \frac{\epsilon}{\sqrt{\epsilon^2 - 4|\eta|^2}}\sinh\sqrt{\epsilon^2 - 4|\eta|^2}\right]\mathbf{a}^\dagger
\end{aligned} \tag{A.12}
$$

APPENDIX B

The Case $\epsilon^2 - 4|\eta|^2 \geq 0$

We introduce $\theta = \sqrt{\epsilon^2 - 4|\eta|^2}$ in (A.9) and (A.12), we obtain

$$e^{\mathbf{A}}\mathbf{a}\,e^{-\mathbf{A}} = \Big[\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big]\mathbf{a} - \frac{2\eta^*}{\theta}\sinh\theta\,\mathbf{a}^\dagger \tag{B.1}$$

$$e^{\mathbf{A}}\mathbf{a}^\dagger e^{-\mathbf{A}} = \frac{2\eta}{\theta}\sinh\theta\,\mathbf{a} + \Big[\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big]\mathbf{a}^\dagger \tag{B.2}$$

Using (B.1) and (B.2) in $\mathbf{H}$, we obtain

$$\begin{aligned}\tilde{\mathbf{H}} = \; & \Big[\omega\Big(\cosh^2\theta - \frac{\epsilon^2 + 4|\eta|^2}{\theta^2}\sinh^2\theta\Big) - 4\alpha\frac{\eta^*}{\theta}\sinh\theta\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) \\ & + 4\beta\frac{\eta}{\theta}\sinh\theta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big)\Big]\mathbf{a}^\dagger\mathbf{a} \\ & + \Big[\omega\Big(\frac{1}{2} - \frac{4|\eta|^2}{\theta^2}\sinh^2\theta\Big) - 2\alpha\frac{\eta^*}{\theta}\sinh\theta\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) \\ & + 2\beta\frac{\eta}{\theta}\sinh\theta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big)\Big] \\ & + \Big[2\omega\frac{\eta}{\theta}\sinh\theta\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) \\ & + \alpha\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big)^2 + 4\beta\frac{\eta^2}{\theta^2}\sinh^2\theta\Big]\mathbf{a}^2 \\ & + \Big[-2\omega\frac{\eta^*}{\theta}\sinh\theta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big) \\ & + 4\alpha\frac{\eta^{*2}}{\theta^2}\sinh^2\theta + \beta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big)^2\Big]\mathbf{a}^{\dagger 2}\end{aligned} \tag{B.3}$$

Using $\cosh^2\theta - \sinh^2\theta = 1$ the equation (B.3) can be written as

$$\begin{aligned}
\tilde{\mathbf{H}} \;=\; & \Big[\omega\Big(1 - \frac{8|\eta|^2}{\theta^2}\sinh^2\theta\Big) - 4\alpha\frac{\eta^*}{\theta}\sinh\theta\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) \\
& +4\beta\frac{\eta}{\theta}\sinh\theta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big)\Big]\Big(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\Big) \\
& + \Big[2\omega\frac{\eta}{\theta}\sinh\theta\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) \\
& + \alpha\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big)^2 + 4\beta\frac{\eta^2}{\theta^2}\sinh^2\theta\Big]\mathbf{a}^2 \\
& + \Big[-2\omega\frac{\eta^*}{\theta}\sinh\theta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big) \\
& + 4\alpha\frac{\eta^{*2}}{\theta^2}\sinh^2\theta + \beta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big)^2\Big]\mathbf{a}^{\dagger 2}
\end{aligned} \tag{B.4}$$

Given that $\tilde{\mathbf{H}}$ is Hermitian, it must requires that; the coefficient of $\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}$ is real and the coefficients of $\mathbf{a}^2$ and $\mathbf{a}^{\dagger 2}$ are respectively complex conjugate one to each other. Using these two requirements, we can have;

$$\begin{aligned}
-4\alpha\frac{\eta^*}{\theta}\sinh\theta\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) + 4\beta\frac{\eta}{\theta}\sinh\theta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big) \\
= \Big\{-4\alpha\frac{\eta^*}{\theta}\sinh\theta\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) \\
+4\beta\frac{\eta}{\theta}\sinh\theta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big)\Big\}^*
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
2\omega\frac{\eta}{\theta}\sinh\theta\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) + \alpha\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big)^2 + 4\beta\frac{\eta^2}{\theta^2}\sinh^2\theta \\
= \Big\{-2\omega\frac{\eta^*}{\theta}\sinh\theta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big) \\
+4\alpha\frac{\eta^{*2}}{\theta^2}\sinh^2\theta + \beta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big)^2\Big\}^*
\end{aligned} \tag{B.6}$$

The equation (B.5) can be written as

$$\frac{\sinh\theta}{\theta}\Big[\alpha\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) + \beta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big)\Big](\eta - \eta^*) = 0 \tag{B.7}$$

The equation (B.7) also yields that

$$\eta - \eta^* = \quad \longrightarrow \quad \eta = \eta^* \tag{B.8}$$

and

$$
\begin{aligned}
\alpha\Big(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\Big) \quad + \quad \beta\Big(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\Big) &= 0 \\
\frac{\epsilon}{\theta}\tanh\theta &= \frac{\alpha+\beta}{\alpha-\beta},
\end{aligned} \tag{B.9}
$$

from which we obtain

$$
\frac{e^{\epsilon} - e^{-\epsilon}}{e^{\epsilon} + e^{-\epsilon}} = \frac{\alpha+\beta}{\alpha-\beta} \quad \longrightarrow \quad -2\beta e^{\epsilon} = 2\alpha e^{-\epsilon} \quad \longrightarrow \quad e^{2\epsilon} = -\frac{\alpha}{\beta} \tag{B.10}
$$

The result in (B.10) shows clearly that (B.9) is inconsistent.
On the other side

$$
\frac{\sinh\theta}{\theta} \to 1 \qquad \longrightarrow \qquad \epsilon^2 - 4|\eta|^2 \to 0 \qquad \longrightarrow \qquad \epsilon \to \pm 2|\eta| \tag{B.11}
$$

This statement in equation (B.11) is somehow offending what we know from [26] by H.B. Geyer, F.G. Scholtz, and I.Snyman. that ϵ and η are independents, But since the transformation **S** remains Hermitian we will also look at this option (B.11).
The only one requirement for (B.5) is that η is real. Let's use this in the equation (B.6), it follows

$$
\begin{aligned}
4\omega\frac{\eta}{\theta}\sinh\theta\cosh\theta + \alpha\Big(\cosh^2\theta + \frac{\epsilon^2 - 4\eta^2}{\theta^2}\sinh^2\theta - \frac{2\epsilon}{\theta}\sinh\theta\cosh\theta\Big) \\
-\beta\Big(\cosh^2\theta + \frac{\epsilon^2 - 4\eta^2}{\theta^2}\sinh^2\theta + \frac{2\epsilon}{\theta}\sinh\theta\cosh\theta\Big) = 0
\end{aligned} \tag{B.12}
$$

As η is real as given in (B.8), it follows that $\theta^2 = \epsilon^2 - 4\eta^2$ and then follows

$$
2\omega\frac{\eta}{\theta}\sinh 2\theta + \alpha\Big(\cosh 2\theta - \frac{\epsilon}{\theta}\sinh 2\theta a\Big) - \beta\Big(\cosh 2\theta + \frac{\epsilon}{\theta}\sinh 2\theta\Big) = 0 \tag{B.13}
$$

which also can be written as

$$
\frac{1}{\theta}\tanh 2\theta = \frac{\alpha - \beta}{(\alpha+\beta)\epsilon - 2\omega\eta} \tag{B.14}
$$

which yields that

$$
\frac{e^{2\theta} - e^{-2\theta}}{e^{2\theta} + e^{-2\theta}} = \frac{(\alpha-\beta)\theta}{(\alpha+\beta)\epsilon - 2\omega\eta} \tag{B.15}
$$

from which follows,

$$\epsilon^{\theta} = \left(\frac{(\alpha + \beta)\epsilon + (\alpha - \beta)\theta - 2\omega\,\eta}{(\alpha + \beta)\epsilon - (\alpha - \beta)\theta - 2\omega\,\eta}\right)^{\frac{1}{4}} \tag{B.16}$$

for $\eta = 0$ in comparison with the result from [26] by H.B. Geyer, F.G. Scholtz and I. Snyman.

$$e^{\epsilon} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \longrightarrow \epsilon = \log\left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \tag{B.17}$$

The Hamiltonian $\tilde{\mathbf{H}}$ can then be written using the requirement η real and the additional requirement stated by the equation (B.6) as,

$$\begin{aligned}\tilde{\mathbf{H}} &= \Big[\omega\left(1 - \frac{8\eta^2}{\theta^2}\sinh^2\theta\right) + 4(\alpha+\beta)\frac{\eta\,\epsilon}{\theta^2}\sinh^2\theta \\ &\quad - 4(\alpha-\beta)\frac{\eta}{\theta}\sinh\theta\,\cosh\theta\Big]\left(\mathbf{a}^{\dagger}\mathbf{a} + \frac{1}{2}\right) \\ &\quad + \Big[-2\omega\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right) + \beta\left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)^2 \\ &\quad + 4\beta\frac{\eta^2}{\theta^2}\sinh^2\theta\Big]\left(\mathbf{a}^2 + \mathbf{a}^{\dagger 2}\right)\end{aligned} \tag{B.18}$$

or

$$\begin{aligned}\tilde{\mathbf{H}} &= \Big[\omega\left(1 - \frac{8\eta^2}{\theta^2}\sinh^2\theta\right) + 4(\alpha+\beta)\frac{\eta\,\epsilon}{\theta^2}\sinh^2\theta \\ &\quad - 4(\alpha-\beta)\frac{\eta}{\theta}\sinh\theta\,\cosh\theta\Big]\left(\mathbf{a}^{\dagger}\mathbf{a} + \frac{1}{2}\right) \\ &\quad + \Big[-2\omega\frac{\eta}{\theta}\sinh\theta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right) + 4\alpha\frac{\eta^2}{\theta^2}\sinh^2\theta \\ &\quad + \beta\left(\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right)^2\Big]\left(\mathbf{a}^2 + \mathbf{a}^{\dagger 2}\right)\end{aligned} \tag{B.19}$$

Where

$$\begin{aligned}\sinh\theta &= \frac{1}{2}\left\{\left[\frac{\beta(\epsilon-\theta) + \alpha(\epsilon+\theta) - 2\omega\eta}{\alpha(\epsilon-\theta) + \beta(\epsilon+\theta) - 2\omega\eta}\right]^{\frac{1}{4}}\right. \\ &\quad \left. - \left[\frac{\alpha(\epsilon-\theta) + \beta(\epsilon+\theta) - 2\omega\eta}{\beta(\epsilon-\theta) + \alpha(\epsilon+\theta) - 2\omega\eta}\right]^{\frac{1}{4}}\right\}\end{aligned} \tag{B.20}$$

and

$$
\begin{aligned}
\cosh\theta &= \frac{1}{2}\left\{\left[\frac{\beta(\epsilon-\theta)+\alpha(\epsilon+\theta)-2\omega\eta}{\alpha(\epsilon-\theta)+\beta(\epsilon+\theta)-2\omega\eta}\right]^{\frac{1}{4}}\right.\\
&\left.+\frac{1}{2}\left[\frac{\alpha(\epsilon-\theta)+\beta(\epsilon+\theta)-2\omega\eta}{\beta(\epsilon-\theta)+\alpha(\epsilon+\theta)-2\omega\eta}\right]^{\frac{1}{4}}\right\} \qquad \text{(B.21)}
\end{aligned}
$$

The Hamiltonians (B.18) and (B.19) which are equivalents can be easily diagonalized using the normal Bogoliubov transformation in which case the spectrum is

$$E_n = (n + \frac{1}{2})\Omega \qquad \text{(B.22)}$$

For $\epsilon^2 - 4|\eta|^2 \to 0$

$$
\begin{aligned}
\tilde{\mathbf{H}} &= \left[\omega(1-2\epsilon^2) + 2\alpha\epsilon(1-\epsilon) + 2\beta\epsilon(1+\epsilon)\right]\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)\\
&+ \left[\omega\,\epsilon(1-\epsilon) + \alpha(1-\epsilon)^2 + \beta\,\epsilon^2\right]\mathbf{a}^2\\
&+ \left[\omega\,\epsilon(1+\epsilon) + \alpha\epsilon^2 + \beta\epsilon(1+\epsilon)^2\right]\mathbf{a}^{\dagger 2} \qquad \text{(B.23)}
\end{aligned}
$$

Where ϵ can be

$$\epsilon = -\frac{1}{2}\frac{\alpha-\beta}{\omega-\alpha-\beta} \qquad \text{(B.24)}$$

or

$$\epsilon = \frac{1}{2}\frac{\alpha-\beta}{\omega+\alpha+\beta} \qquad \text{(B.25)}$$

APPENDIX C

C.1 Evaluation of $\lambda^{(0)}_{mn}$ for $\eta = 0$

We consider the matrix element

$$\lambda^{(0)}_{mn} = \langle n_b|\mathbf{B}|m\rangle,$$

We know that the unity operator in position representation is

$$I = \int_{-\infty}^{\infty} dx|x\rangle\langle x| \tag{C.1}$$

Therefore, λ_{mn} becomes

$$\lambda^{(0)}_{mn} = \int_{-\infty}^{\infty} dx\langle n_b|\mathbf{B}|x\rangle\langle x|m\rangle. \tag{C.2}$$

Since

$$\tilde{\mathbf{H}}\mathbf{B}|n_b\rangle = \left(n + \frac{1}{2}\right)\sqrt{\omega^2 - 4\alpha\beta}\,\mathbf{B}|n_b\rangle \tag{C.3}$$

$$\begin{aligned}\langle n_b|\mathbf{B}|x\rangle &= \Psi^*_n\left([\omega^2 - 4\alpha\beta]^{\frac{1}{4}}x\right)\\ &= \frac{1}{2^n n!\sqrt{\pi}}e^{-\frac{1}{2}\sqrt{\omega^2-4\alpha\beta}x^2}H_n(\Omega^{\frac{1}{2}}x)\end{aligned} \tag{C.4}$$

where the $H_n(\Omega^{\frac{1}{2}}x)$ are the Hermite's functions and $\Omega = \sqrt{\omega^2 - 4\alpha\beta}$. And

$$\mathbf{H} = \left(\mathbf{a}^{\dagger}\mathbf{a} + \frac{1}{2}\right)\omega \tag{C.5}$$

$$\mathbf{H}|m\rangle = \left(m + \frac{1}{2}\right)\omega\,|m\rangle \tag{C.6}$$

$$\begin{aligned}\langle x|m\rangle &= \Psi_m\left(\omega^{\frac{1}{2}}x\right)\\ &= \frac{1}{2^m m!\sqrt{\pi}}e^{-\frac{1}{2}\omega x^2}H_m\left(\omega^{\frac{1}{2}}x\right)\end{aligned} \tag{C.7}$$

It follows

$$\lambda^{(0)}_{mn} = \frac{1}{2^{m+n}m!\,n!\,\pi}\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)x^2}H_n(\Omega^{\frac{1}{2}}x)H_m(\omega^{\frac{1}{2}}x) \tag{C.8}$$

Therefore

$$\lambda^{(0)}_{mn} = \begin{cases} 0 \quad For\ m+n\ odd \\ \\ \left[\frac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)^{m+n+1}}\right]^{\frac{1}{2}} \quad For\ m+n\ even \end{cases} \tag{C.9}$$

C.2 Evaluation of w_{nm} for $\eta = 0$

$$
\begin{aligned}
w_{nm} &= \langle m|\mathbf{T}e^{-i\mathbf{H}\tau}|n\rangle_R \\
&= \langle m|e^{-i\mathbf{H}^\dagger\tau}\mathbf{T}|n\rangle_R \\
&= \langle m|e^{-i\mathbf{H}^\dagger\tau}|n\rangle_L \\
&= e^{-iE_n\tau}\langle m|\mathbf{S}\mathbf{B}^\dagger|n_b\rangle
\end{aligned}
\tag{C.10}
$$

For $\eta = 0$, the similarity transformation becomes $\mathbf{S} = \left(\frac{\alpha}{\beta}\right)^{\hat{\mathbf{n}}/4}$. which in (C.10) gives

$$w_{nm} = \left[\frac{\alpha}{\beta}\right]^{\frac{m}{4}} e^{-iE_n t}\lambda^{(0)}_{nm} \tag{C.11}$$

where

$$\lambda^{(0)}_{nm} = \frac{1}{2^{m+n}m!\,n!\,\pi}\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)} H_m(\omega^{\frac{1}{2}}x)H_n([\omega^2-4\alpha\beta]^{\frac{1}{4}}x) \tag{C.12}$$

Therefore

$$I_{nm} = \begin{cases} 0 \quad For\ m+n\ odd \\ \\ \left[\dfrac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)^{m+n+1}}\right]^{\frac{1}{2}} \quad For\ m+n\ even \end{cases} \tag{C.13}$$

which gives

$$w_{nm} = \begin{cases} 0 \quad For\ \ m+n\ \ odd \\ \\ \left(\frac{\alpha}{\beta}\right)^{\frac{m}{2}}\left[\dfrac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)^{m+n+1}}\right]^{\frac{1}{2}} e^{-iE_n t} \quad For\ \ m+n\ \ even \end{cases} \tag{C.14}$$

The transition takes place only from the states $|n\rangle_R$ to the states $|m\rangle$ fulfilling the condition $m+n$ *even*, with the probability

$$|w_{nm}|^2 = 2\left[\frac{\alpha}{\beta}\right]^{\frac{m}{2}} \frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)^{m+n+1}} \tag{C.15}$$

C.3 Evaluation of w_{mn} and w_{nm} for $\eta = \frac{\epsilon}{2}$

We consider

$$w_{mn} = e^{-iE_n\tau}\langle n_b|\mathbf{B}e^{\epsilon\hat{\mathbf{x}}^2}|m\rangle. \tag{C.16}$$

We make use of the operator unity (C.1), and obtain

$$
\begin{aligned}
w_{mn} &= e^{-iE_n t}\langle n_b|\mathbf{B}e^{\epsilon\hat{\mathbf{x}}^2}|m\rangle \\
&= e^{-iE_n t}\int_{-\infty}^{\infty} dx\langle n_b|\mathbf{B}e^{\epsilon\hat{\mathbf{x}}^2}|x\rangle\langle x|m\rangle \\
&= e^{-iE_n t}\int_{-\infty}^{\infty} dx e^{\epsilon x^2}\langle n_b|\mathbf{B}|x\rangle\langle x|m\rangle
\end{aligned} \tag{C.17}
$$

From which we define

$$
\begin{aligned}
\lambda_{mn}^{(\frac{\epsilon}{2})} &= \frac{1}{2^{m+n}m!\,n!\,\pi}\int_{-\infty}^{\infty} dx\Big[e^{-\frac{1}{2}\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)x^2}e^{\epsilon x^2} \\
&\quad H_n(\Omega^{\frac{1}{2}}x)H_m(\omega^{\frac{1}{2}}x)\Big]
\end{aligned} \tag{C.18}
$$

Therefore

$$
\lambda_{mn}^{(\frac{\epsilon}{2})} = \begin{cases} 0 \quad \textit{For } m+n \textit{ odd} \\ \\ \left[\dfrac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}-2\epsilon\right)^{m+n+1}}\right]^{\frac{1}{2}} \quad \textit{For } m+n \textit{ even} \end{cases} \tag{C.19}
$$

Therefore the probability of transition becomes

$$
|w_{mn}|^2 = 2\frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}-\frac{\alpha-\beta}{\omega-\alpha-\beta}\right)^{m+n+1}} \tag{C.20}
$$

We consider the reverse transition taking from the eigenstates $|n\rangle_R$ to the states $|m\rangle$. Using the specific expression of $\mathbf{S}$ for $\eta = \frac{\epsilon}{2}$, and introducing the unity operator $\mathbf{I}$ (C.1) we obtain

$$
\begin{aligned}
w_{nm} &= e^{-iE_n t}\int_{-\infty}^{\infty} dx\langle m|e^{\epsilon\,\hat{\mathbf{x}}^2}|x\rangle\langle x|\mathbf{B}^\dagger|n_b\rangle \\
&= e^{-iE_n t}\lambda_{nm}
\end{aligned} \tag{C.21}
$$

Which gives

$$
\begin{aligned}
\lambda_{nm}^{(\frac{\epsilon}{2})} &= \frac{1}{2^{m+n}m!\,n!\,\pi}\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}\left(\omega+\sqrt{\omega^2-4\alpha\beta}\right)x^2}e^{\frac{\alpha-\beta}{2(\omega-\alpha-\beta)}x^2} \\
&\quad H_m(\omega^{\frac{1}{2}}x)H_n(\Omega^{\frac{1}{2}}x)
\end{aligned} \tag{C.22}
$$

Therefore

$$\lambda_{nm}^{(\frac{\epsilon}{2})} = \begin{cases} 0 \quad For\ m+n\ odd \\ \\ \left[\frac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}-\frac{\alpha-\beta}{\omega-\alpha-\beta}\right)^{m+n+1}}\right]^{\frac{1}{2}} \quad For\ m+n\ even \end{cases} \tag{C.23}$$

which gives

$$w_{nm} = \begin{cases} 0 \quad For \quad m+n \quad odd \\ \\ \left[\frac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}-\frac{\alpha-\beta}{\omega-\alpha-\beta}\right)^{m+n+1}}\right]^{\frac{1}{2}} e^{-iE_n t} \quad For \quad m+n \quad even \end{cases} \tag{C.24}$$

The transition takes place only from the states $|n\rangle_R$ to the states $|m\rangle$ fulfilling the condition $m+n \quad even$, with the probability

$$|w_{nm}|^2 = 2\frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}-\frac{\alpha-\beta}{\omega-\alpha-\beta}\right)^{m+n+1}} \tag{C.25}$$

C.4 Evaluation of w_{mn} and w_{nm} for $\eta = -\frac{\epsilon}{2}$

For the last case, we consider the case where $\eta = -\frac{\epsilon}{2}$, the similarity transformation is $\mathbf{S} = e^{\epsilon \hat{\mathbf{p}}^2}$. Which substituted in (C.10) gives

$$w_{mn} = e^{-iE_n\tau}\langle n_b|\mathbf{B}e^{\epsilon\hat{\mathbf{p}}^2}|m\rangle. \tag{C.26}$$

We introduce the operator unity $\mathbf{I}$ in momentum representation

$$\mathbf{I} = \int_{-\infty}^{\infty} dp|p\rangle\langle p|. \tag{C.27}$$

We introduce the unity operator (C.27) in (C.26) and obtain

$$\begin{aligned} w_{mn} &= e^{-iE_n t}\langle n_b|\mathbf{B}e^{\epsilon\hat{\mathbf{p}}^2}|m\rangle \\ &= e^{-iE_n t}\int_{-\infty}^{\infty} dp\langle n_b|\mathbf{B}e^{\epsilon\hat{\mathbf{x}}^2}|p\rangle\langle p|m\rangle \\ &= e^{-iE_n t}\int_{-\infty}^{\infty} dp e^{\epsilon p^2}\langle n_b|\mathbf{B}|p\rangle\langle p|m\rangle \end{aligned} \tag{C.28}$$

From which we define

$$\lambda_{mn}^{(-\frac{\epsilon}{2})} = \frac{1}{2^{m+n} m!\, n!\, \pi} \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}\left(\omega + \sqrt{\omega^2 - 4\alpha\beta}\right) p^2} e^{\epsilon p^2} H_n(\Omega^{\frac{1}{2}} p) H_m(\omega^{\frac{1}{2}} p) \tag{C.29}$$

Therefore

$$\lambda_{mn}^{(-\frac{\epsilon}{2})} = \begin{cases} 0 \quad For\ m+n\ odd \\ \left[\frac{2\omega^m \left(\omega^2 - 4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega + \sqrt{\omega^2 - 4\alpha\beta} - 2\epsilon\right)^{m+n+1}}\right]^{\frac{1}{2}} \quad For\ m+n\ even \end{cases} \tag{C.30}$$

On the other hand, we know that for $\eta = -\frac{\epsilon}{2},\ \epsilon = -\frac{\alpha-\beta}{2(\omega+\alpha+\beta)}$, which gives

$$w_{mn} = \begin{cases} 0 \quad For \quad m+n \quad odd \\ \left[\frac{2\omega^m \left(\omega^2 - 4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega + \sqrt{\omega^2 - 4\alpha\beta} + \frac{\alpha-\beta}{(\omega+\alpha+\beta)}\right)^{m+n+1}}\right]^{\frac{1}{2}} e^{-iE_n t} \quad For \quad m+n \quad even \end{cases} \tag{C.31}$$

The transition takes place only from the states $|m\rangle$ to the states $|n\rangle_R$ fulfilling the condition $m+n \quad even$, with the probability

$$|w_{mn}|^2 = 2 \frac{\omega^m \left(\omega^2 - 4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega + \sqrt{\omega^2 - 4\alpha\beta} + \frac{\alpha-\beta}{\omega+\alpha+\beta}\right)^{m+n+1}} \tag{C.32}$$

We consider the reverse transition taking from the eigenstates $|n\rangle_R$ to the states $|m\rangle$. Using the specific expression of $\mathbf{S}$ for $\eta = -\frac{\epsilon}{2}$, and introducing the unity operator $\mathbf{I}$ (C.27) we obtain

$$\begin{aligned} w_{nm} &= e^{-iE_n t} \int_{-\infty}^{\infty} dp \langle m | e^{\epsilon\, \hat{\mathbf{p}}^2} | p \rangle \langle p | \mathbf{B}^{\dagger} | n_b \rangle \\ &= e^{-iE_n t} \lambda_{nm} \end{aligned} \tag{C.33}$$

Which gives

$$\begin{aligned} \lambda_{nm} = \frac{1}{2^{m+n} m!\, n!\, \pi} \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}\left(\omega + \sqrt{\omega^2 - 4\alpha\beta}\right) p^2} e^{\frac{\alpha-\beta}{2(\omega-\alpha-\beta)} p^2} \\ H_m(\omega^{\frac{1}{2}} p) H_n(\Omega^{\frac{1}{2}} p) \end{aligned} \tag{C.34}$$

Therefore

$$\lambda_{nm} = \begin{cases} 0 \quad For\ m+n\ odd \\ \\ \left[\frac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}+\frac{\alpha-\beta}{\omega+\alpha+\beta}\right)^{m+n+1}}\right]^{\frac{1}{2}} \quad For\ m+n\ even \end{cases} \tag{C.35}$$

which gives

$$w_{nm} = \begin{cases} 0 \quad For \quad m+n \quad odd \\ \\ \left[\frac{2\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}+\frac{\alpha-\beta}{\omega+\alpha+\beta}\right)^{m+n+1}}\right]^{\frac{1}{2}} e^{-iE_n t} \quad For \quad m+n \quad even \end{cases} \tag{C.36}$$

The transition takes place only from the states $|n\rangle_R$ to the states $|m\rangle$ fulfilling the condition $m+n$ *even*, with the probability

$$|w_{nm}|^2 = 2\frac{\omega^m\left(\omega^2-4\alpha\beta\right)^{\frac{n}{2}}}{\left(\omega+\sqrt{\omega^2-4\alpha\beta}+\frac{\alpha-\beta}{\omega+\alpha+\beta}\right)^{m+n+1}} \tag{C.37}$$

APPENDIX D

D.1 Determination of the g_i 's for $\hat{\mathbf{n}}$ commuting with the metric $\mathbf{U}$

Let us consider the requirement (4.57)

$$\mathbf{U}\hat{\mathbf{n}} = \hat{\mathbf{n}}\mathbf{U}.$$

We recall from (3.18) the quasi-particles creation operator $\mathbf{c}$, and the quasi-particle destruction operator $\mathbf{d}$. We substitute $\mathbf{a}$ and $\mathbf{a}^\dagger$ in (4.57) by the corresponding expression with respect to $\mathbf{c}$ and $\mathbf{d}$ (3.23). We obtain

$$\begin{aligned} \mathbf{U}\mathbf{a}^\dagger\mathbf{a} &= \mathbf{a}^\dagger\mathbf{a}\mathbf{U} \\ \mathbf{U}(g_3\mathbf{d} + g_4\mathbf{c})(g_1\mathbf{d} + g_2\mathbf{c}) &= \mathbf{a}^\dagger\mathbf{a}\mathbf{U}. \end{aligned} \tag{D.1}$$

Using the Swanson's expressions [25]((41a) and (41b) page 591) we can derive

$$\begin{aligned} (g_3\mathbf{c}^\dagger + g_4\mathbf{d}^\dagger)\mathbf{U}(g_1\mathbf{d} + g_2\mathbf{c}) &= \mathbf{a}^\dagger\mathbf{a}\mathbf{U} \\ (g_3\mathbf{c}^\dagger + g_4\mathbf{d}^\dagger)(g_1\mathbf{c}^\dagger + g_2\mathbf{d}^\dagger)\mathbf{U} &= \mathbf{a}^\dagger\mathbf{a}\mathbf{U}. \end{aligned} \tag{D.2}$$

Which reduces to

$$\left(g_3\mathbf{c}^\dagger + g_4\mathbf{d}^\dagger\right)\left(g_1\mathbf{c}^\dagger + g_2\mathbf{d}^\dagger\right) = \mathbf{a}^\dagger\mathbf{a}. \tag{D.3}$$

Which also can be written as,

$$\begin{aligned} &\left[g_3(-g_3^*\mathbf{a}^\dagger + g_1^*\mathbf{a}) + g_4(g_4^*\mathbf{a}^\dagger - g_2^*\mathbf{a})\right] \\ &\left[g_1(-g_3^*\mathbf{a}^\dagger + g_1^*\mathbf{a}) + g_2(g_4^*\mathbf{a}^\dagger - g_2^*\mathbf{a})\right] = \mathbf{a}^\dagger\mathbf{a} \\ &\left[(g_4g_4^* - g_3g_3^*)\mathbf{a}^\dagger + (g_1^*g_3 - g_2^*g_4)\mathbf{a}\right] \\ &\left[(g_2g_4^* - g_1g_3^*)\mathbf{a}^\dagger + (g_1g_1^* - g_2g_2^*)\mathbf{a}\right] = \mathbf{a}^\dagger\mathbf{a}. \end{aligned} \tag{D.4}$$

Identifying the left-hand side and the right-hand side of (D.4), it follows that

$$g_1^*g_3 - g_2^*g_4 = 0 \tag{D.5}$$

$$g_2g_4^* - g_1g_3^* = 0 \tag{D.6}$$

$$(g_1g_1^* - g_2g_2^*)(g_4g_4^* - g_3g_3^*) = 1. \tag{D.7}$$

Using (D.5) or (D.6), we deduce

$$\frac{g_1}{g_2} = \frac{g_4^*}{g_3^*}, \tag{D.8}$$

or

$$\frac{g_1^*}{g_2^*} = \frac{g_4}{g_3}. \tag{D.9}$$

Using the canonical condition $g_1 g_4 - g_2 g_3 = 1$ and (D.7), we can set up the following system of two equations.

$$\begin{cases} g_1 g_4 - g_2 g_3 = 1 \\ \\ g_1 g_3^* - g_2 g_4^* = 0 \end{cases} \tag{D.10}$$

Multiplying the first equation of (D.10) by g_3^* and the second equation of (D.10) by $-g_4$ and summing them, we obtain

$$|g_4|^2 - |g_3|^2 = \frac{g_3^*}{g_2} \tag{D.11}$$

For all $g_j = \rho_j e^{i\varphi_j}$ where $j = 1, 2, 3, 4$, the equation (D.11) implies

$$\varphi_3 = -\varphi_2 \tag{D.12}$$

Multiplying the first equation of (D.10) by g_4^* and the second equation of (D.10) by $-g_3$ and summing them we obtain

$$|g_4|^2 - |g_3|^2 = \frac{g_4^*}{g_1} \tag{D.13}$$

which correspondingly leads to

$$\varphi_4 = -\varphi_1 \tag{D.14}$$

Using the Swanson's results $g_1 g_2 = -\frac{\beta}{\sqrt{\omega^2 - 4\alpha\beta}}$ and $g_3 g_4 = -\frac{\alpha}{\sqrt{\omega^2 - 4\alpha\beta}}$ [25] page 597, we obtain

$$\begin{aligned} \frac{g_4 g_3}{g_1 g_2} &= \frac{\alpha}{\beta} \\ \frac{\rho_4 \rho_3}{\rho_1 \rho_2} &= \frac{\alpha}{\beta} \\ \frac{\rho_3}{\rho_2} &= \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}. \end{aligned} \tag{D.15}$$

Combining (D.10) and (D.11) we obtain

$$\frac{g_3^*}{g_2} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} \tag{D.16}$$

Multiplying side by side (D.14) and the Swanson's result $g_2 g_3 = \frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}$ [25] (page 597 equation (B7)), we obtain

$$
\begin{aligned}
\rho_3^2 &= \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} \frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}} \\
\rho_3 &= \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \left[\frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}}.
\end{aligned}
\tag{D.17}
$$

Substituting ρ_3 in (D.16), we obtain

$$
\rho_2 = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}} \left[\frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}}
\tag{D.18}
$$

Using the Swanson's expression $g_1 g_4 = \frac{2\alpha\beta}{\sqrt{\omega^2 - 4\alpha\beta}(\omega - \sqrt{\omega^2 - 4\alpha\beta})}$, we can similarly obtain

$$
\rho_4 = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \left[\frac{\omega + \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}}
\tag{D.19}
$$

$$
\rho_1 = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}} \left[\frac{\omega + \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}}
\tag{D.20}
$$

The $g_1 g_2 = -\frac{\beta}{\sqrt{\omega^2 - 4\alpha\beta}}$ shows that

$$
\varphi_2 = -\varphi_1
\tag{D.21}
$$

Therefore all the g_j s are explicitly obtained

$$
\begin{cases}
g_1 = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}} \left[\frac{\omega + \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}} e^{i\varphi} \\
g_2 = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}} \left[\frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}} e^{-i\varphi} \\
g_3 = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \left[\frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}} e^{i\varphi} \\
g_4 = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \left[\frac{\omega + \sqrt{\omega^2 - 4\alpha\beta}}{2\sqrt{\omega^2 - 4\alpha\beta}}\right]^{\frac{1}{2}} e^{-i\varphi}
\end{cases}
\tag{D.22}
$$

D.2 Determination of the g_i 's for $\hat{\mathbf{x}}$ commuting with the metric $\mathbf{U}$

$$
\begin{aligned}
\mathbf{U}\hat{\mathbf{x}} &= \hat{\mathbf{x}}\mathbf{U} \\
\mathbf{U}(\mathbf{a} + \mathbf{a}^\dagger) &= (\mathbf{a} + \mathbf{a}^\dagger)\mathbf{U}
\end{aligned}
\tag{D.23}
$$

We substitute $\mathbf{a}$ and $\mathbf{a}^\dagger$ in (D.23) by the corresponding expression with respect to $\mathbf{c}$ and $\mathbf{d}$. We obtain

$$
\begin{aligned}
\mathbf{U}\left[(g_1\mathbf{d} + g_2\mathbf{c}) + (g_3\mathbf{d} + g_4\mathbf{c})\right] &= \left(\mathbf{a} + \mathbf{a}^\dagger\right)\mathbf{U} \\
\left[(g_1 + g_3)\mathbf{c}^\dagger + (g_2 + g_4)\mathbf{d}^\dagger)\right]\mathbf{U} &= \left(\mathbf{a} + \mathbf{a}^\dagger\right)\mathbf{U}
\end{aligned}
\tag{D.24}
$$

Which gives

$$
(g_1 + g_3)\mathbf{c}^\dagger + (g_2 + g_4)\mathbf{d}^\dagger) = \mathbf{a} + \mathbf{a}^\dagger \tag{D.25}
$$

$$
(g_1 + g_3)(g_1^*\mathbf{a} - g_3^*\mathbf{a}^\dagger) + (g_2 + g_4)(-g_2^*\mathbf{a} + g_4^*\mathbf{a}^\dagger) = \mathbf{a} + \mathbf{a}^\dagger \tag{D.26}
$$

The comparison between the coefficients on both (D.26) sides yields the system of two linear equation with respect to $(g_1 + g_3)$ and $(g_2 + g_4)$

$$
\begin{cases}
g_1^*(g_1 + g_3) - g_2^*(g_2 + g_4) = 1 \\
\\
-g_3^*(g_1 + g_3) + g_4^*(g_2 + g_4) = 1
\end{cases}
\tag{D.27}
$$

Which solved gives;

$$
(g_1 + g_3) = (g_2^* + g_4^*) \tag{D.28}
$$

We substitute the result (D.28) in the first equation of the system (D.27), it follows

$$
g_1^*(g_1 + g_3) - g_2^*(g_1^* + g_3^*) = 1. \tag{D.29}
$$

Let $z = g_1 + g_3$, the equation (D.29) becomes

$$
g_1^* z - g_2^* z^* = 1, \tag{D.30}
$$

where z is a complex variable $z = x + iy$, we deduce

$$
(g_1^* - g_2^*)x = 1 \tag{D.31}
$$

$$
(g_1^* - g_2^*)y = 0 \tag{D.32}
$$

which yields $y = 0$, and consequently

$$
g_1 + g_3 = g_1^* + g_3^* \tag{D.33}
$$

Therefore

$$\begin{aligned} g_1 + g_3 &= \frac{1}{g_1 - g_2} \\ g_3 &= -g_1 + \frac{1}{g_1 - g_2} \end{aligned} \tag{D.34}$$

Using the result (D.34) in (D.28), and substituting g_3 from (D.34), we obtain

$$g_1 + g_3 = g_2 + g_4 \tag{D.35}$$

$$g_4 = -g_2 + \frac{1}{g_1 - g_2} \tag{D.36}$$

we divide (D.34) by g_1 and we use the Swanson's result $\frac{g_3}{g_1} = -\frac{\omega - \sqrt{\omega^2-4\alpha\beta}}{2\beta}$ [25] page 597, it gives

$$\frac{1}{g_1^2 - g_1 g_2} = 1 - \frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\beta} \tag{D.37}$$

Using the Swanson's result $g_1 g_2 = -\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}}$ [25] page 597, we derive

$$g_1^2 = -\frac{\beta}{\sqrt{\omega^2 - 4\alpha\beta}} + \left[1 - \frac{\omega - \sqrt{\omega^2 - 4\alpha\beta}}{2\beta}\right]^{-1} \tag{D.38}$$

Therefore all the g_j s are explicitly obtained

$$\begin{cases} g_1 = \left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1 - \frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{\frac{1}{2}} \\ g_2 = -\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1 - \frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{-\frac{1}{2}} \\ g_3 = -\frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1 - \frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{\frac{1}{2}} \\ g_4 = \frac{\omega+\sqrt{\omega^2-4\alpha\beta}}{2\sqrt{\omega^2-4\alpha\beta}}\left(-\frac{\beta}{\sqrt{\omega^2-4\alpha\beta}} + \left[1 - \frac{\omega-\sqrt{\omega^2-4\alpha\beta}}{2\beta}\right]^{-1}\right)^{-\frac{1}{2}} \end{cases} \tag{D.39}$$

APPENDIX E

We consider the local form of the similarity transformation $\mathbf{S}$ (2.9) and (2.9)

$$\begin{cases} \mathbf{S}\mathbf{a}\mathbf{S}^{-1} = \left[\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right]\mathbf{a} - \frac{2\eta}{\theta}\sinh\theta\mathbf{a}^{\dagger} \\ \\ \mathbf{S}\mathbf{a}^{\dagger}\mathbf{S}^{-1} = \frac{2\eta}{\theta}\sinh\theta\mathbf{a} + \left[\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right]\mathbf{a}^{\dagger} \end{cases}$$

from which we deduce the following transformations:

$$\mathbf{S}\hat{\mathbf{x}}\mathbf{S}^{-1} = \cosh\theta\hat{\mathbf{x}} - \frac{i}{\omega}\frac{\epsilon-2\eta}{\theta}\sinh\theta\hat{\mathbf{p}} \tag{E.1}$$

$$\mathbf{S}\hat{\mathbf{p}}\mathbf{S}^{-1} = \cosh\theta\hat{\mathbf{p}} + i\omega\frac{\epsilon+2\eta}{\theta}\sinh\theta\hat{\mathbf{x}} \tag{E.2}$$

Conversely we also have

$$\begin{cases} \mathbf{S}^{-1}\mathbf{a}\mathbf{S} = \left[\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta\right]\mathbf{a} + \frac{2\eta}{\theta}\sinh\theta\,\mathbf{a}^{\dagger} \\ \\ \mathbf{S}^{-1}\mathbf{a}^{\dagger}\mathbf{S} = -\frac{2\eta}{\theta}\sinh\theta\,\mathbf{a} + \left[\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right]\mathbf{a}^{\dagger} \end{cases} \tag{E.3}$$

from which we deduce the following transformations:

$$\mathbf{S}^{-1}\hat{\mathbf{x}}\mathbf{S} = \cosh\theta\hat{\mathbf{x}} + \frac{i}{\omega}\frac{\epsilon-2\eta}{\theta}\sinh\theta\hat{\mathbf{p}} \tag{E.4}$$

$$\mathbf{S}^{-1}\hat{\mathbf{p}}\mathbf{S} = \cosh\theta\hat{\mathbf{p}} - i\omega\frac{\epsilon+2\eta}{\theta}\sinh\theta\hat{\mathbf{x}} \tag{E.5}$$

We multiply side by side the requiremet $\mathbf{x}\mathbf{T} = \mathbf{T}\mathbf{x}^{\dagger}$ (5.32) first by $\mathbf{S}^{-1}$ from the left and next from the right. which gives,

$$\mathbf{S}^{-1}\mathbf{x}\mathbf{S} = \mathbf{S}\mathbf{x}^{\dagger}\mathbf{S}^{-1} \tag{E.6}$$

We apply the previous results (E.1),(E.2),(E.3), and (E.4) to (E.5) using the expression of $\mathbf{x}$ in (5.30)

$$\begin{aligned} \mathbf{S}^{-1}\Big(\sqrt{\frac{\omega}{\Omega}}\cosh\theta\hat{\mathbf{x}} + \frac{i}{\sqrt{\omega\Omega}}\frac{\epsilon-2\eta}{\theta}\sinh\theta\hat{\mathbf{p}}\Big)\mathbf{S} &= \mathbf{S}\Big(\sqrt{\frac{\omega}{\Omega}}\cosh\theta\hat{\mathbf{x}} - \frac{i}{\sqrt{\omega\Omega}}\frac{\epsilon-2\eta}{\theta}\sinh\theta\hat{\mathbf{p}}\Big)\mathbf{S}^{-1} \\ \sqrt{\frac{\omega}{\Omega}}\hat{\mathbf{x}} + \frac{i}{\sqrt{\omega\Omega}}\frac{\epsilon-2\eta}{\theta}\sinh 2\theta\hat{\mathbf{p}} &= \sqrt{\frac{\omega}{\Omega}}\hat{\mathbf{x}} - \frac{i}{\sqrt{\omega\Omega}}\frac{\epsilon-2\eta}{\theta}\sinh 2\theta\hat{\mathbf{p}} \\ \frac{2i}{\sqrt{\omega\Omega}}\frac{\epsilon-2\eta}{\theta}\sinh 2\theta\hat{\mathbf{p}} &= 0 \end{aligned} \tag{E.7}$$

The equation (E.7) is only satisfied for $\eta = \frac{\epsilon}{2}$.

And for the the requiremet $\mathbf{pT} = \mathbf{Tp}^{\dagger}$ (5.33) first by $\mathbf{S}^{-1}$ from the left and next from the right. which gives,

$$\mathbf{S}^{-1}\mathbf{x}\mathbf{S} = \mathbf{S}\mathbf{x}^{\dagger}\mathbf{S}^{-1} \tag{E.8}$$

We apply the previous results (E.1),(E.2),(E.3), and (E.4) to (E.8) using the expression of $\mathbf{p}$ in (5.31)

$$\begin{aligned}
\mathbf{S}^{-1}\Big(\sqrt{\frac{\Omega}{\omega}}\cosh\theta\hat{\mathbf{p}} - i\sqrt{\omega\Omega}\frac{\epsilon+2\eta}{\theta}\sinh\theta\hat{\mathbf{x}}\Big)\mathbf{S} &= \mathbf{S}\Big(\sqrt{\frac{\Omega}{\omega}}\cosh\theta\hat{\mathbf{p}} + i\sqrt{\omega\Omega}\frac{\epsilon+2\eta}{\theta}\sinh\theta\hat{\mathbf{x}}\Big)\mathbf{S}^{-1} \\
\sqrt{\frac{\Omega}{\omega}}\hat{\mathbf{p}} - i\sqrt{\omega\Omega}\frac{\epsilon+2\eta}{\theta}\sinh 2\theta\hat{\mathbf{x}} &= \sqrt{\frac{\Omega}{\omega}}\hat{\mathbf{x}} - i\sqrt{\omega\Omega}\frac{\epsilon+2\eta}{\theta}\sinh 2\theta\hat{\mathbf{x}} \\
2i\sqrt{\omega\Omega}\frac{\epsilon+2\eta}{\theta}\sinh 2\theta\hat{\mathbf{x}} &= 0
\end{aligned} \tag{E.9}$$

The equation (E.9) is only satisfied for $\eta = -\frac{\epsilon}{2}$.

BIBLIOGRAPHY

[1] F.G. Scholtz, H.B. Geyer, and F. J. W. Hahne, Annals of Physics **213** 74 (1992)

[2] Carl M. Bender, and Stefan Boettcher, Physical Review Letters **80** 24 (1998) 5243-5246

[3] Carl M. Bender, Stefan Boettcher, Peter N. Meisinger, Journal of Mathematical Physics **40** 5 (1999) 2201-2229

[4] Carl M. Bender, arXiv:quant-ph/0501052v1 (2006)

[5] Carl M. Bender, Jun-Hua Chen, and Kimball A. Milton, Journal of Physics A: Mathematical and General **39** 5 (2006) page 1657-1668

[6] Miloslav Znojil Physics Letter A **259** 3-4 (2001) 220-223

[7] Q. Duret, and F. Gieres www1.jinr.ru/Proceedings/Burdik-2004

[8] Hagen Keinert, PARTH INTEGRALS in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, *World Scientific* (2004) 12-30

[9] Mostafazadeh A. Batal A, Journal of Physics A: Mathematical and General **37** 48 (2004) 11645-11679

[10] Mostafazadeh A, Journal of Mathematical Physics **43** 1 (2002) 205

[11] Mostafazadeh A, Journal of Mathematical Physics **43** 5 (2002) 2814

[12] Mostafazadeh A. and Seher O, arXiv:quant-ph/0607120v1 (2006)

[13] Michael Reed and Barry Simon, Functional Analysis, *Academic Press* (1980)

[14] Kreyszig E, Introductory Functional Analysis With Applications, *John Wilson & Sons* (1989)

[15] Sadri Hassani, Mathematical Physics A Modern Introduction to Its Foundations, *Springer* (1998)

[16] Raphael A. Lehrer and David R. Nelson, Physical Review B **58** 12 (1998) 385

[17] L. Mandel and E. Wolf, Optical coherence and Quantum Optics, *Cambridge University Press* (1996)

[18] C. Cohen-Tannoudji, J. Dupont-Roc and Gilbert Grynberg, Introduction to Quantum Electrodynamics *John Wiley and Sons* (1989)

[19] Freeman J. Dyson, Physical Review **110** 2 (1958) pages 579-581

[20] W.D. Heiss, Journal of Physics A: Mathematical and General **37** 6 (2004) 2455-2454

[21] C. Cohen-Tannoudji, Jacques Dupont-Roc, Gilbert Grynberg, Photons and Atoms Introduction to Quantum Electrodynamics *John Wilson & Sons* (1989).

[22] Ian Affleck, Walter Hofstetter, David R. Nelson, and Ulrich Schollwock, arXiv:cond-mat/0408478 v2 (2004)

[23] J.Rogel-Salazar, G.H.C. New, S Choi, and K Burnett, Journal of Physics B **34** 23 (2001) 4617-4627)

[24] Alexender L. Fetter, arXiv:cond-math/9811366 v1 **26** (1998)

[25] Mark S. Swanson, Journal of Mathematical Physics **45** 2 (2004) 585-601

[26] H.B. Geyer, F.G. Scholtz, Izak Snyman, Czechoslovak Journal of Physics **54** 10 (2004) 1069-1073

[27] F.G. Scholtz, H.B. Geyer, Journal of Physics A: Mathematical and General **39** (2006) 10189

[28] H. F. Jones, Journal of Physics A: Mathematical and General **38** (2005) 1741-1746

[29] Tosio Kato, Perturbation theory for linear operators, *Springer-Verlag* (1966)

[30] Richard L Liboff, Introductory Quantum Mechanics *Addison-Wesley* Second edition (1993)

[31] O Bartteli, and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, *Springer-Verlag, New York* **1, 2** (1987)

[32] Adrian Heathcote, Philosophy of Science **57** 3 (1990) 523-534

[33] D. Janssen, P. Schuk, Nuclear Physics A **501** (1989) 270-288